Extra Credit Project Solutions

Basic Integrals

1) $\int x^5 dx = \frac{x^6}{6} + C$

2) $\int 4x + 3\,dx = 2x^2 + 3x + C$

3) $\int_0^4 (1+3y-y^2)\,dy = (y + \frac{3}{2}y^2 - \frac{y^3}{3})\Big|_0^4 = \frac{20}{3}$

4) $\int_1^2 \frac{3}{t^4}\,dt = -\frac{3}{3}\frac{1}{t^3}\Big|_1^2 = \frac{7}{8}$

5) $\int \frac{x^2+1}{\sqrt{x}}\,dx = \frac{2}{5}x^{5/2} + 2\sqrt{x} + C$

6) $\int_0^{\pi/2} \sin u\,du = 1$

7) $\int_1^8 \frac{1}{\sqrt[3]{x}} = \frac{3}{2}x^{2/3}\Big|_1^8 = \frac{9}{2}$

8) $\int u(\sqrt{u} + \sqrt[3]{u})\,du = \frac{2}{5}u^{5/2} + \frac{3}{7}u^{7/3} + C$

9) $\int_0^5 2e^x + 4\cos x\,dx = (2e^x + 4\sin x)\Big|_0^5 = 2e^5 + 4\sin 5 - 2$

10) $\int_{\pi/4}^{\pi/2} \csc\theta\tan\theta\,d\theta = -\csc\theta\Big|_{\pi/4}^{\pi/2} = -(1-\sqrt{2}) = \sqrt{2}-1$

11) $\int (1-u)(2+u^2)\,du = \int -u^3 + u^2 - 2u + 2\,du = -\frac{u^4}{4} + \frac{u^3}{3} - u^2 + 2u + C$

u-substitution (easy)

1) $\int \sin(3\theta)\,d\theta = -\frac{\cos(3\theta)}{3} + C$; $u = 3\theta$

2) $\int x e^{x^2} dx = \frac{1}{2}e^{x^2} + C$; $u = x^2$

3) $\int_0^1 \cos(\pi t)\,dt = \frac{\sin(\pi t)}{\pi}\Big|_0^1 = 0$; $u = \pi t$

4) $\int_0^{\pi/4} \sin(4t)\,dt = -\frac{\cos(4\pi)}{4}\Big|_0^{\pi/4} = \frac{1}{2}$; $u = 4t$

5) $\int 2x(x^2+3)^4\,dx = \frac{(x^2+3)^5}{5} + C$; $u = x^2+3$

6) $\int x^3(1-x^4)^5\,dx = -\frac{(1-x^4)^6}{24} + C$; $u = 1-x^4$

7) $\int (2-x)^6\,dx = -\frac{(2-x)^7}{7} + C$; $u = 2-x$

8) $\int \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2} + C$; $u = x-1$

9) $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1| + C$; $u = x^2+1$

10) $\int \frac{1}{5-3x}\,dx = -\frac{1}{3}\ln|5-3x| + C$; $u = 5-3x$

11) $\int \frac{4x+1}{\sqrt{1+x+2x^2}}\,dx = 2(1+x+2x^2)^{\frac{1}{2}} + C$; $u = 2x^2+x+1$

12) $\int \sqrt[3]{3-5y}\,dy = -\frac{3}{20}(3-5y)^{4/3} + C$; $u = 3-5y$

13) $\int \frac{2}{(1+u)^6}\,du = -\frac{2}{5}\,\frac{1}{(1+u)^5} + C$; $w = 1+u$

u-substitution (harder)

1) $\int \frac{1}{\sqrt{x}} \sin(\sqrt{x})\,dx = -2\cos\sqrt{x} + C$; $u = \sqrt{x}$

2) $\int_1^4 \frac{1}{\sqrt{x}} e^{\sqrt{x}}\,dx = 2e^{\sqrt{x}}\Big|_1^4 = 2e(e-1)$; $u = \sqrt{x}$

3) $\int \cos(x)\, e^{\sin x}\,dx = e^{\sin x} + C$; $u = \sin x$

4) $\int e^x \sqrt{1+e^x}\,dx = \frac{2}{3}(1+e^x)^{3/2} + C$; $u = 1+e^x$

5) $\int \tan\theta\,d\theta = -\ln|\cos\theta| + C$; $u = \cos\theta$

6) $\int \cot\theta\,d\theta = \ln|\sin\theta| + C$; $u = \sin\theta$

7) $\int \sin^3\theta\,d\theta = \frac{\cos^3\theta}{3} - \cos\theta + C$; $u = \cos\theta$

8) $\int \frac{1}{x}[\ln(x)]^2\,dx = \frac{[\ln(x)]^3}{3} + C$; $u = \ln(x)$

9) $\int \frac{\tan^{-1}x}{1+x^2}\,dx = \frac{(\tan^{-1}x)^2}{2} + C$; $u = \tan^{-1}x$

10) $\int \cos^2\theta\,d\theta = \frac{1}{2}\int 1+\cos(2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$; $u = 2\theta$

11) $\int \sin^2\theta\,d\theta = \int 1-\cos^2\theta\,d\theta = \theta - \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$

12) $\int \sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\int \sin^2 2\theta\,d\theta = \frac{1}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right) + C$

13) $\int \cos^5 x\,dx = \int \cos^4 x\,d(\sin x) = \int (1-\sin^2 x)^2\,d(\sin x) = \frac{\sin^5 x}{5} - \frac{2\sin^3 x}{3} + \sin x + C$; $u = \sin x$

14) $\int \tan^2\theta\sec^4\theta\,d\theta = \int \tan^2\theta\sec^2\theta\,d(\tan\theta) = \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3} + C$; $u = \tan\theta$

Integration by parts

1) $\int \theta\cos\theta\,d\theta = \theta\sin\theta + \cos\theta + C$; $u = \theta$; $dv = \cos\theta\,d\theta$

2) $\int x^2\sin(ax)\,dx = -\frac{1}{a}x^2\cos ax + \frac{2}{a^2}x\sin ax + \frac{2}{a^3}\cos ax + C$; by part twice: 1) $u = x^2$; $dv = \sin ax\,dx$ 2) $u = x$; $dv = \cos ax\,dx$

3) $\int (x^2+1)e^{-x}\,dx = \int x^2e^{-x} + e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 3e^{-x} + C$; 1) $u = x^2$; $dv = e^{-x}dx$ 2) $u = x$; $dv = e^{-x}dx$

4) $\int t^3e^t\,dt = t^3e^t - 3t^2e^t + 6te^t - 6e^t + C$; 1) $u = t^3$; $dv = e^t dt$ 2) $u = t^2$; $dv = e^t dt$ 3) $u = t$; $dv = e^t dt$

5) $\int \ln(x)\,dx = x\ln x - x + C$; $u = \ln(x)$; $dv = 1\,dx$

6) $\int x^5\ln x\,dx = \frac{1}{6}x^6\ln(x) - \frac{1}{36}x^6 + C$; $u = \ln(x)$; $dv = x^5dx$

7) $\int \sqrt{t}\,\ln(t)\,dt = \frac{2}{3}t^{3/2}\ln t - \frac{4}{9}t^{3/2} + C$; $u = \ln(t)$; $dv = \sqrt{t}\,dt$

8) $\int x^{3/2}\ln(x)\,dx = \frac{2}{5}x^{5/2}\ln(x) - \frac{4}{25}x^{5/2} + C$; $u = \ln(x)$; $dv = x^{3/2}dx$

9) $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$; $u = \sin^{-1}(x)$; $dv = dx$, then u-sub : $w = 1-x^2$

10) $\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$; $u = \tan^{-1}x$; $dv = x\,dx$

11) $\int \sin[\ln(t)]\,dt = t\sin[\ln t] - \int \cancel{t}\cos(\ln t)/\cancel{t}\,dt = t\sin(\ln t) - t\cos(\ln t) - \int \sin(\ln t)\,dt$

$u = \sin(\ln t)$, $dv = dt$; $u = \cos(\ln t)$, $dv = dt$

$\Rightarrow \int \sin(\ln t)\,dt = \frac{1}{2}t(\sin(\ln t) - \cos(\ln t)) + C$

12) $\int e^{-\theta}\cos(2\theta)\,d\theta = -e^{-\theta}\cos 2\theta - \int 2e^{-\theta}\sin 2\theta\,d\theta = -e^{-\theta}\cos 2\theta + 2\left(e^{-\theta}\sin 2\theta - 2\int e^{-\theta}\cos 2\theta\,d\theta\right)$

$\Rightarrow \int e^{-\theta}\cos 2\theta\,d\theta = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta$ (Typical!)

13) $\int x^5\cos(x^3)\,dx = \int x^3\cos(x^3)\,x^2\,dx = \frac{1}{3}\int w\cos w\,dw$ $(w = x^3)$

$= \frac{1}{3}[w\sin w + \cos w] + C$; $u = w$; $dv = \cos w\,dw$

$= \frac{1}{3}[x^3\sin x^3 + \cos x^3] + C$; remember to plug $w = x^3$ back!

14) $\int x^5 e^{x^3}\,dx = \frac{1}{3}\int x^3 e^{x^3} x^2\,dx = \frac{1}{3}(x^3 e^{x^3} - e^{x^3}) + C$ (same logic as # 13)

Partial fractions

1) $\int \frac{x}{x-4}\,dx = \int \frac{(x-4)+4}{(x-4)}\,dx = \int 1 + \frac{4}{x-4}\,dx = x + 4\ln|x-4| + C$

2) $\int \frac{1}{(t+4)(t-1)}\,dt = \frac{1}{5}\int \frac{1}{t-1} - \frac{1}{t+4}\,dt = \frac{1}{5}\{\ln|t-1| - \ln|t+4|\} + C$

3) $\int \frac{x^3+x^2-12x+1}{x^2+x-12}\,dx = \int x + \frac{1}{(x+4)(x-3)}\,dx = \int x - \frac{1}{7}\frac{1}{(x+4)} + \frac{1}{7}\frac{1}{(x-3)}\,dx$

$= \frac{x^2}{2} + \frac{1}{7}\{\ln|x-3| - \ln|x+4|\} + C$

4) $\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$\Rightarrow x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$

Plug $x = 3 \Rightarrow 9 = A(3+2)^2 \Rightarrow A = \frac{9}{25}$

Plug $x = -2 \Rightarrow 4 = C(-2-3) \Rightarrow C = -\frac{4}{5}$

Plug $x = 0 \Rightarrow \frac{9}{25}(2)^2 + B(-6) + (-\frac{4}{5})(-3) = 0 \Rightarrow B = \frac{16}{25}$

$\Rightarrow \int \frac{x^2}{(x-3)(x+2)^2}\,dx = \frac{9}{25}\int \frac{1}{x-3}\,dx + \frac{16}{25}\int \frac{1}{x+2}\,dx - \frac{4}{5}\int \frac{1}{(x+2)^2}\,dx$

$= \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5}\frac{1}{(x+2)} + C$

5) $\frac{2x^2+5}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$

$\Rightarrow 2x^2+5 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1) = (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D)$

$\Rightarrow A+C = 0$; $4A+C = 0 \Rightarrow A = C = 0$

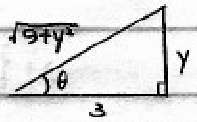
& $\left.\begin{array}{l} B+D = 2 \\ 4B+D = 5 \end{array}\right\} \Rightarrow \begin{array}{l} B = 1 \\ D = 1 \end{array}$

$\Rightarrow \int \frac{2x^2+5}{(x^2+1)(x^2+4)}\,dx = \int \frac{1}{x^2+1}\,dx + \int \frac{1}{x^2+4}\,dx = \tan^{-1}x + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

Trig. Substitution

1) $\int \frac{1}{\sqrt{9+y^2}}\,dy$ let $y = 3\tan\theta$; $dy = 3\sec^2\theta\,d\theta$

$= \int \frac{3\sec^2\theta}{3\sec\theta}\,d\theta = \int \sec\theta\,d\theta$



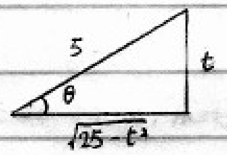
$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C$

2) $\int \sqrt{25-t^2}\,dt$ let $t = 5\sin\theta$; $dt = 5\cos\theta$

$= \int 25\cos^2\theta\,d\theta$

$= \frac{25}{2}\int 1+\cos 2\theta\,d\theta$



$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{25}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

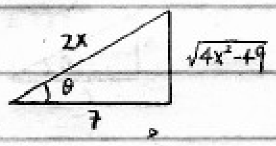
$= \frac{25}{2}\left(\arcsin\left(\frac{t}{5}\right) + \frac{t}{5}\cdot\frac{\sqrt{25-t^2}}{5}\right) + C$

$= \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{1}{2}\sqrt{25-t^2} + C$

3) $\int \frac{1}{\sqrt{4x^2-49}}\,dx$ let $x = \frac{7}{2}\sec\theta$; $dx = \frac{7}{2}\sec\theta\tan\theta\,d\theta$

$= \int \frac{1}{7\tan\theta}\cdot\frac{7}{2}\sec\theta\tan\theta\,d\theta$

$= \frac{1}{2}\int \sec\theta\,d\theta$



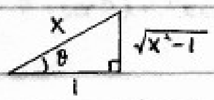
$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$

$= \frac{1}{2}\ln\left|\frac{2}{7}x + \frac{\sqrt{4x^2-49}}{7}\right| + C$

4) $\int (x^2-1)^{-3/2}\,dx$ let $x = \sec\theta$; $dx = \sec\theta\tan\theta\,d\theta$

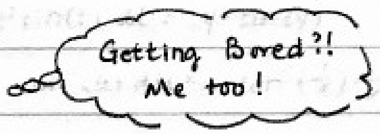
$= \int (\tan^2\theta)^{-3/2}\cdot\sec\theta\tan\theta\,d\theta$

$= \int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$



$= -\frac{1}{\sin\theta} + C$

$= -\frac{x}{\sqrt{x^2-1}} + C$



5) $\int \frac{x^3}{\sqrt{16-x^2}}\,dx$ $\qquad x = 4\sin\theta \ldots$

$= \frac{64}{4}\cdot 4\int \frac{\sin^3\theta}{\cos\theta}\cos\theta\,d\theta$ $\qquad$ you know how to do it, right ?

$= -64\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$

$= -64\left\{\frac{\sqrt{16-x^2}}{4} - \frac{1}{3}\cdot\frac{1}{64}\cdot(16-x^2)\cdot\sqrt{16-x^2}\right\} + C$

$= -\frac{1}{3}\sqrt{16-x^2}\left\{48 - 16 + x^2\right\} + C$

$= -\frac{1}{3}\sqrt{16-x^2}\,(32+x^2) + C$

6) $\int \frac{8}{(4x^2+1)^2}\,dx$ $\qquad 2x = \tan\theta \;;\; dx = \frac{1}{2}\sec^2\theta\,d\theta$

$= 8\int \frac{1}{\sec^4\theta}\cdot\frac{1}{2}\sec^2\theta\,d\theta$

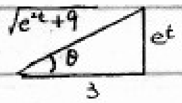
$= 4\int \cos^2\theta\,d\theta$ $\qquad$ ($\int\cos^2\theta\,d\theta$ again, make sure you know how to do it !)

$= 4\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$

$= 2\tan^{-1}(2x) + \frac{x}{1+4x^2} + C$

7) $\int \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$ $\qquad$ let $e^t = 3\tan\theta \;;\; e^t dt = 3\sec^2\theta\,d\theta$

$= \int \frac{1}{\sec\theta}\sec^2\theta\,d\theta$



$= \int \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{e^{2t}+9}}{3} + \frac{e^t}{3}\right| + C$

Don't Forget Integrals

1) $\int \sec\theta\,d\theta$: $\qquad \left.\begin{array}{l} d(\tan\theta) = \sec^2\theta\,d\theta \\ d(\sec\theta) = \sec\theta\tan\theta\,d\theta \end{array}\right\} \Rightarrow d(\tan\theta + \sec\theta) = \sec\theta(\sec\theta + \tan\theta)\,d\theta$

$\Rightarrow \int \frac{1}{\sec\theta + \tan\theta}\,d(\sec\theta + \tan\theta) = \int \sec\theta\,d\theta$

$\Rightarrow \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$

2) $\frac{d}{dx}\int_1^x \ln u\,du = \ln x$ $\qquad$ (use F.T.C.)

3) $\frac{d}{dy}\int_2^y t^2\sin(t)\,dt = y^2\sin(y)$

4) $\frac{d}{dx}\int_2^{\frac{1}{x}} \tan^{-1}(u)\,du = -\tan^{-1}\left(\frac{1}{x}\right)\frac{1}{x^2}$