6. INTEGRAL CALCULUS

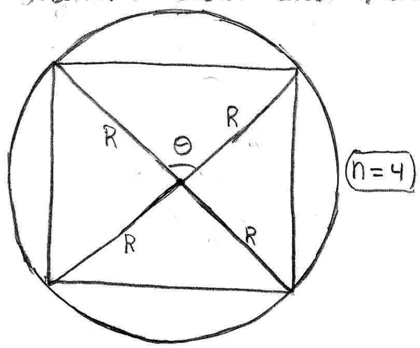
In the differential calculus we began by studying how to use limits to define the tangent line to a graph. We then defined the derivative, saw how to calculate the derivatives of basic elementary functions, then their sums, products, quotients and composites. After that we saw how logarithmic differentiation allowed us to compute the derivatives of inverse functions. Having established the calculational foundation we proceeded to study how to apply derivatives to a number of interesting applications. Mainly we have seen that the derivative describes how a function changes with respect to its input variable. This in turn gave us much geometric information about the graph, particularly increase/decrease and concavity.

We now change gears a bit, a different question motivates integral calculus. How can we calculate the area of a curvy shape? This question is ancient. The Greeks and Chinese thought of dividing a given curvy region into smaller shapes which have a well-known area. The area of a rectangle is length times width, this defines area. Then for a shape more complicated than a rectangle we can imagine filling it with lots of little rectangles and then the area of the shape would be the sum of the areas of the rectangles. This method required a large investment of arithmetic. The definition of area has not changed, in fact intuitively we still agree with the ancients. What is new is what Isaac Barrow discovered and Newton, Leibniz and countless others developed. They learned that the derivative contains information about the area in a somewhat indirect manner. The integral and derivative are said to be “inverse processes”. This idea is made precise in the Fundamental Theorem of Calculus (FTC). The FTC will allow us to find the so-called “signed-area” under a curve. The full resolution of the area problem will have to wait until the next chapter.

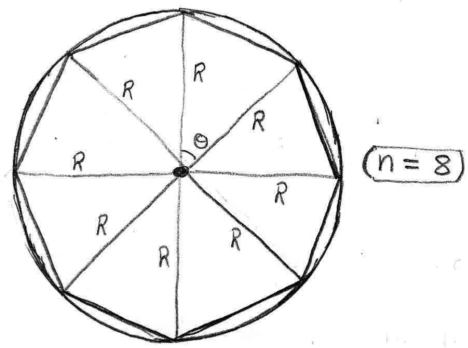
Integral calculus need not much harder than differential calculus. However, you will find it is nearly impossible unless you already have a firm grasp on the differential calculus. If you have uncertainty on doing the basic derivatives I suggest you remedy that before getting too far into this chapter. When I say it is not much harder that is directed to the student, I am careful to choose those problems which have simple solutions. In contrast to differential calculus we will see that it is very easy to come up with functions which have integrals such that formula in terms of elementary functions exists.

The example that follows next is rather unusual, I just want to illustrate how you can find areas even without the FTC. There is something subtle here, we have never proved that 3.1415… really is the ratio of the circumference and the diameter. There are proofs of π, but I don’t think its in Stewart or my notes at the moment. So when we prove that $A = \pi R^2$ its not a complete proof, we ignore the question of motivating the value for π itself.

• Lets study a rather unconventional example before we dive into the usual integration theory. We consider inscribing a regular n-sided polygon inside a circle of radius R. I illustrate the case $n=4$ (square) and $n=8$ (octogon) we then generalize to the n^{th} case and take $n \to \infty$ to fine the area of circle,

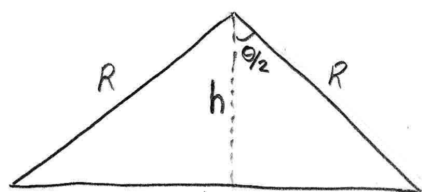


$\theta = \frac{2\pi}{n} = \frac{\pi}{2}$



$\theta = \frac{2\pi}{n} = \frac{\pi}{4}$

Lets use trigonometry to calculate the area of a typical subtriangle. Need to find b and h using R and θ.



$$b = 2R\sin(\theta/2)$$
$$h = R\cos(\theta/2)$$

$$A = \frac{1}{2}bh = R^2\sin(\theta/2)\cos(\theta/2)$$

Now you should know $\sin(2\beta) = 2\sin\beta\cos\beta$ thus

$$A = \frac{1}{2}R^2\sin(\theta)$$

Lets write the area of a regular n-gon inscribed in the circle of radius R

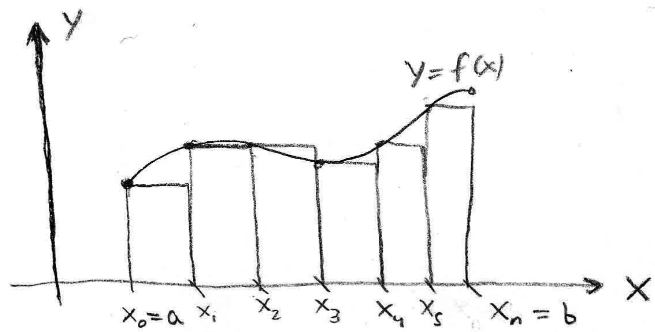
$$A_n = n \cdot \frac{1}{2}R^2\sin(\theta) = \boxed{\frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right) = A_n}$$

For finite n we approximate the area of circle, but as $n \to \infty$ this becomes an exact account of the circle's area,

$$\begin{aligned} A_\infty &= \lim_{n\to\infty}\left(\frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)\right) \\ &= \pi R^2 \lim_{n\to\infty}\left(\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}\right) \\ &= \pi R^2 \lim_{t\to 0^+}\left(\frac{\sin(t)}{t}\right) = \boxed{\pi R^2} \end{aligned}$$

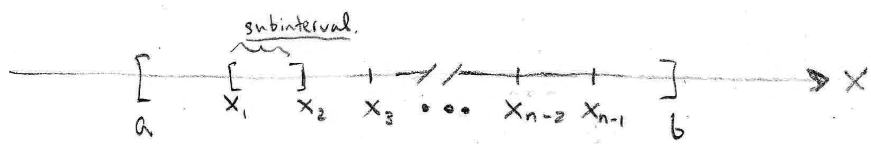
6.1. AREA UNDER A CURVE

We can approximate the area under the curve $y = f(x)$ by dividing the shape up into a bunch of rectangles. The more rectangles we use the better the approximation of the area. For example, here is an illustration of the approximation of the area under $y = f(x)$ for $a \leq x \leq b$ where the left endpoint of the rectangle is used to set the height of the approximating rectangle.



We use boxes or equal length. To find that length we take the total length $b - a$ and divide by the number of approximating rectangle n. There will be n-subintervals.

$$
\begin{aligned}
x_0 &= a \\
x_1 &= a + \Delta x \\
x_2 &= a + 2\Delta x \\
&\vdots \\
x_n &= a + n\Delta x
\end{aligned}
$$



The illustration used left-endpoints to determine the height of the rectangles but there are four other choices that we need to mention here:

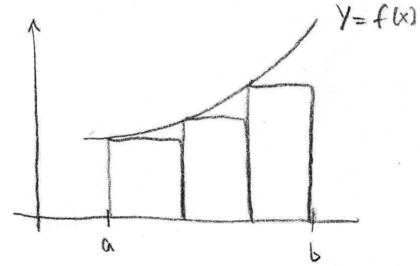
I. Left endpoint rule (L_n)
II. Right endpoint rule (R_n)
III. Mid-point rule (M_n)
IV. Riemann sum, uses arbitrary point as sample point $(\mathcal{R}_n)$

We will give formulas for the first three cases. The fourth case we will use to define the definite integral, the freedom to choose any point in each subinterval is important to our proof of the FTC. It can be shown that as $n \to \infty$ these various approximation schemes will converge to a single value, the area under the curve $y = f(x)$ from $x = a$ to $x = b$. This "area" can be negative if $f(x) < 0$ for some $x \in [a, b]$. So we will refer to what we are calculating as the "signed-area".

I.) Left Endpoint Rule with n-boxes : L_n.

$$\boxed{L_n = \Delta x\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right] = \sum_{i=0}^{n-1} f(x_i)\Delta x}$$

Geometrically:



L_3 underestimates area, since f inc. on (a,b)

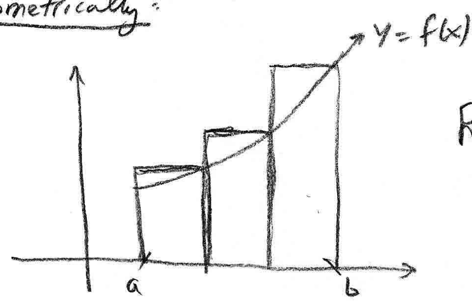
Remark: • If $f(x)$ is increasing on $(a,b) \Rightarrow L_n$ underestimates area
• If $f(x)$ is decreasing on $(a,b) \Rightarrow L_n$ overestimates area.

Reminder: We can find where $f(x)$ is inc/dec by examining $f'(x)$; $f'(x) > 0 \ \forall x \in (a,b) \Rightarrow f$ inc. on (a,b)
$f'(x) < 0 \ \forall x \in (a,b) \Rightarrow f$ dec. on (a,b)

II.) Right Endpt Rule with n-boxes : R_n

$$\boxed{R_n = \Delta x\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right] = \sum_{i=1}^{n} f(x_i)\Delta x}$$

Geometrically:



R_3 overestimates area since f inc. on (a,b)

Remark: its geometrically clear that
• If $f(x)$ inc. on $(a,b) \Rightarrow R_n$ overestimates area.
• If $f(x)$ dec on $(a,b) \Rightarrow R_n$ underestimates area.

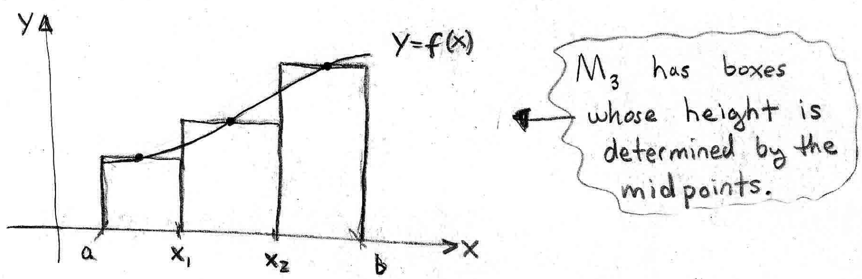
I'll let you calculate a couple of these in the homework. The midpoint gives the best estimate in many cases, but the precise estimation of error is too involved for this course. I encourage you to take numerical methods if that sort of question is intriguing.

III. Midpoint Rule with n-boxes : M_n

Define the n-midpoints : $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ to be the average of the endpoints of each sub-interval $[x_i, x_{i+1}]$ of $[a, b]$,

$$\overline{x}_1 = \tfrac{1}{2}(x_0 + x_1)$$
$$\overline{x}_2 = \tfrac{1}{2}(x_1 + x_2)$$
$$\vdots$$
$$\overline{x}_i = \tfrac{1}{2}(x_{i-1} + x_i)$$
$$\vdots$$
$$\overline{x}_n = \tfrac{1}{2}(x_{n-1} + x_n)$$

$$\boxed{M_n = \Delta x\left[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\right] = \sum_{i=1}^{n} f(\overline{x}_i)\,\Delta x}$$



IV. Riemann sum with n-boxes : $\mathcal{R}_n$

Pick $x_i^* \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$ thru whatever method you like,

$$\mathcal{R}_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

this is the Riemann sum, we'll use it to define the definite integral shortly. (The fact that x_i^* is arbitraily chosen is important to the proof of the fundamental theorem of calculus.)

Remark: L_n, R_n, M_n and $\mathcal{R}_n$ will give the area under $y = f(x)$ from $x = a$ to $x = b$ if we let n become very large. That is,

$$\text{AREA} = \lim_{n\to\infty}(L_n) = \lim_{n\to\infty}(R_n) = \lim_{n\to\infty}(M_n) = \lim_{n\to\infty}(\mathcal{R}_n)$$

We can calculate these sums for any function which is continuous. In fact, we could even calculate these for a function which was discontinuous at a finite number of points. If you think about it we can just add together the area under each piece of a piecewise-defined function.

__Definition6.1.1: (Definite Integral; Riemann Integral)__
Let a function f be continuous at all but a finite number of points on the interval $[a, b]$ then we define the definite integral of f from a to b as follows:

$$\int_a^b f(x)dx \equiv \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i{}^*)\triangle x \quad \text{where} \quad \triangle x = \frac{b-a}{n}.$$

The sample point $x_i{}^* \in [x_{i-1}, x_i]$, but we make no particular restriction on the choice, we can use any point in the subinterval provided it has $f(x_i^*) \in \mathbb{R}$ (avoid discontinuities). The function $f(x)$ is called the *__integrand__* and the value a is the *__lower bound or limit of integration__* while b is the *__upper bound or limit of integration__*. We call dx the

Notice that this definition allows us to use the left, right or mid-point rules if we are asked to "calculate the integral from the definition". This is still a daunting task for even simple functions.

__Example 6.1.1:__ *(not an efficient method to calculate area!)*

E1

$$\int_0^3 (x+2)dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{3i}{n}+2\right)\frac{3}{n}\right) \quad : \quad x_0 = 0,\ x_1 = \frac{3}{n}, \ldots x_i = \frac{3i}{n}, \ldots, x_n = 3\frac{n}{n}$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{9i}{n^2} + \sum_{i=1}^{n}\frac{6}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{9}{n^2}\sum_{i=1}^{n} i + \frac{1}{n}\sum_{i=1}^{n} 6\right) \quad : \ n \text{ indep. of } i$$

$$= \lim_{n\to\infty}\left(\frac{9}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n}\cdot 6n\right)$$

$$= \frac{9}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right) + 6 \quad : \ \text{prop. of limits}$$

$$= \frac{9}{2}\lim_{n\to\infty}\left(\frac{1+1/n}{1}\right) + 6 \quad : \ \text{dividing by } n$$

$$= \boxed{\frac{9}{2} + 6}$$

__Remark__: These formulas are friends if you are asked to find an integral directly from the def$^{\underline{n}}$.

4 $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	6 $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
5 $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$	7 $\displaystyle\sum_{i=1}^{n} c = nc$

Notice that the width of the rectangle $\triangle x = \frac{b-a}{n}$ goes to zero as the number of approximating rectangles goes to infinity. Intuitively, $\triangle x \to dx$ and $\Sigma \to \int$. The integral is an infinite sum, Leibniz used $\int$ to enforce the notion that the integral is a summation. Technically, dx is not a number while $\triangle x$ is a number, it is finite while dx is known as an *infinitesimal.* We will find that infinitesimal arguments provide an essential tool in the application of integration to real-world problems. I'm getting a little ahead in the story, let's get back to the basics.

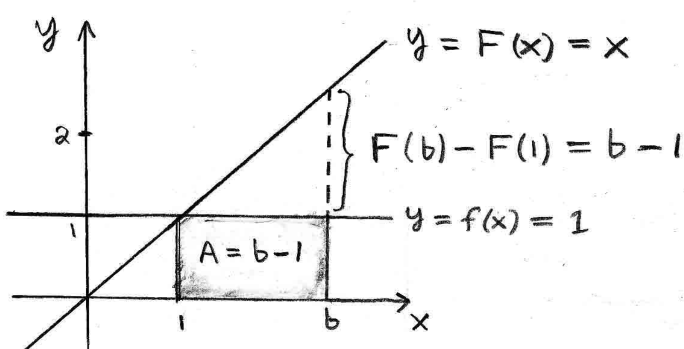
Definition6.1.2: (Indefinite Integral; antiderivatives)
Let a function f be continuous at all but a finite number of points on some subset $U \subseteq \mathbb{R}$ then we define the indefinite integral of f on U as:

$$\int f(x)dx \equiv F(x) \quad \text{where} \quad \frac{dF}{dx} = f(x)$$

where $F(x)$ is a family of functions which includes every function $G(x)$ such that $\frac{dF}{dx} = \frac{dG}{dx}$ on U. We say that $G(x)$ is an antiderivative of $f(x)$ if $\frac{dG}{dx} = f(x)$. The function $f(x)$ is called the ***integrand*** and the integral is called *indefinite* because it is not definite, it has no upper and lower limits. We also call dx the measure of integration in this context.

I should mention that the family of functions $F(x)$ is the most general antiderivative. We may carelessly refer to F as a function, but strictly speaking this is incorrect, $\int f(x)dx$ is actually a whole family of functions. For example, if $f(x) = 1$ then $\int 1dx = x + c$ while we could say $F(x) = x + 2$ is a particular antiderivative. Since the whole family of functions have equal derivatives it follows that they differ by at most a constant by Theorem 5.3.6.

At this stage you would rightly chastise me for calling this an "integral" after all what does this indefinite integral have to do with area? Let's focus on a really easy example. Let $y = f(x) = 1$ let's consider the area under the curve on $[1, b]$. An antiderivative for $f(x) = 1$ is $F(x) = x$. Look at the graphs,



You can see that the change in the antiderivative over the interval gave us the area under the curve in this case.

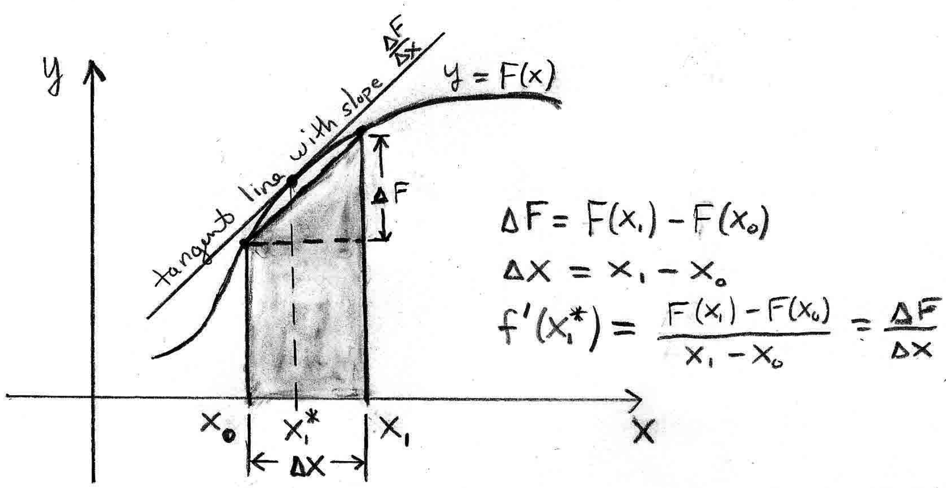
Can we generalize this example? What if anything can we say in general about areas and tangent lines? In particular we should think about the tangent line to the graph of the antiderivative. What will this tell us about the function? Is there some way to pick the height of the approximating rectangle that uses the derivative of the antiderivative on the interval? Recall the Mean Value Theorem for $y = F(x)$ on the interval $[x_0, x_1]$ tells us that there exists $c \in [x_0, x_1]$ such that

$$F'(c) = \frac{F(x_0) - F(x_1)}{x_1 - x_0} = \frac{F(x_0) - F(x_1)}{\triangle x}$$

We can choose the sample point to be this point; $x_1^* = c$. Since F is the antiderivative of f we find $f(x_1{}^*) = (F(x_0) - F(x_1))/\triangle x$

$$f(x_1{}^*) = (F(x_0) - F(x_1))/\triangle x \implies \boxed{f(x_1{}^*)\triangle x = F(x_0) - F(x_1).}$$

Here's a picture of how the Mean Value Theorem works for F.



Remark: I suppose this makes sense to look at the Mean Value Theorem, it does give us a link between the derivative and the values of the function. Of course the values of the function will go toward what the area is under the curve. So, if the function f is the derivative of another function F then it stands to reason the derivatives of F should tell us about the area under f. And more than that, the derivatives of F are related to the values of F and hence the area. A more direct argument to try to begin is to use $\frac{df}{dx}$, after all the derivative is related to the function through the mean value Theorem. However, the way in which they are related is not helpful in calculating the definite integral.

Goal: find a nice formula for the definite integral that uses the antiderivative.

Proof of the Fundamental Theorem of Calculus: (we use the discussion above to motivate our choice of sample points in the Riemann integral)

We will pick the sample points in the Riemann integral below to be those points such that the Mean Value Theorem is satisfied for the antiderivative on the given subinterval. That is we insist that $x_i^* \in [x_i, x_{i-1}]$ such that $f(x_i{}^*)\triangle x = F(x_{i-1}) - F(x_i)$. We are free to choose such points because the antiderivative is by definition differentiable and continuous. Thus the Mean Value Theorem applies to F.

$$\begin{aligned}
\int_a^b f(x)\ dx &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\ \triangle x \right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} \big[F(x_i) - F(x_{i-1})\big] \right) \\
&= \lim_{n\to\infty}\left(\big[F(x_1) - F(x_0) + F(x_2) - F(x_1) + \cdots + F(x_n) - F(x_{n-1})\big] \right) \\
&= \lim_{n\to\infty}\left(\big[-F(x_0) + F(x_n)\big] \right) \\
&= \lim_{n\to\infty}\left(\big[-F(a) + F(b)\big] \right) \\
&= F(b) - F(a).
\end{aligned}$$

This concludes the proof of the Fundamental Theorem of Calculus (FTC). Lets state the result:

> *Theorem 6.1.1:* (Fundamental Theorem of Calculus) If f is continuous on $[a,b]$ with antiderivative $F(x)$ then
>
> $$\boxed{\int_a^b f(x)\ dx = F(b) - F(a).}$$

This result clearly extends to piecewise continuous functions. We can apply the FTC to each piece and take the sum of those results. This Theorem is amazing. We can calculate the area under a curve based on the values of the antiderivative at the endpoints. Think about that, if $a = 1$ and $b = 3$ then $\int_1^3 f(x)dx$ depends only on $F(3)$ and $F(1)$. Doesn't it seem intuitively likely that what value $f(2)$ takes should matter as well? Why don't we have to care about $F(2)$? The values of the function at $x = 2$ certainly went into the calculation of the area, if we calculate a left sum we would need to take values of the function between the endpoints. The cancellation that occurs in the proof is the root of why my naïve intuition is bogus.

Remark: the variable inside the integral is known as a "dummy variable of integration". We can write another letter in the place just the same. Overall this integral is not a function of x, rather it is a number. We can write:

$$\int_a^b f(x)\ dx = \int_a^b f(t)\ dt = \int_a^b f(u)\ du = F(b) - F(a).$$

In contrast we cannot say the same for the indefinite integral. When we write $\int f(x)\ dx$ we have a family of functions of x, on the other hand $\int f(t)\ dt$ that would a family of functions of t.

6.2. INDEFINITE INTEGRATION

In this section we discuss all the elementary integrals. We find these by educated guessing. We know the derivatives, we just have to go backwards and tweak it a bit here or there.

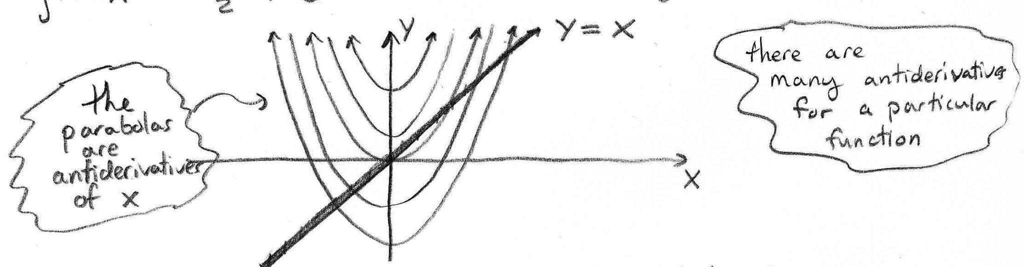
Example 6.2.1: *(here we discuss the meaning of "c")*

E1 Let $f(x) = x$ notice that there are many antiderivatives of x:

$F_1(x) = \frac{1}{2}x^2$ is antiderivative of f because $\frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$

$F_2(x) = \frac{1}{2}x^2 + 3$ is also an a.d. of f because $\frac{d}{dx}\left(\frac{1}{2}x^2 + 3\right) = x$

$\int x\,dx = \frac{x^2}{2} + C$ is the indefinite integral of x.



The value of C is left arbitrary so that the indefinite integral is the general antiderivative (it includes all possible antiderivatives like F_1 or F_2 ...)

Example 6.2.2:

E2
$$\begin{aligned}\int (3e^x + 2)\,dx &= \int 3e^x dx + \int 2\,dx \\ &= 3\int e^x dx + 2\int dx \\ &= 3e^x + C_1 + 2x + C_2 \quad : \text{differentiate and you'll get back } e^x \text{ and } 1. \\ &= \boxed{3e^x + 2x + C}\end{aligned}$$

Lets check our answer,

$$\frac{d}{dx}\left(3e^x + 2x + C\right) = 3e^x + 2$$

The derivative of the antiderivative is the integrand, it checks.

Example 6.2.3:

$\boxed{E3}$

$$\int \cosh(x)\,dx = \sinh(x) + C$$

$$\int \sinh(x)\,dx = \cosh(x) + C$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

the signs are 'opposite' of those of differentiation

How did I know the above integrals? Simple, I know how to differentiate, so I just have to think backwards. In each case I ask myself what function can I differentiate to get the integrand.

Summary of all the basic integrals including a few you are not expected to memorize in Calculus I:

$f(x)$	$\int f(x)\,dx - C$
1	x
x	$\frac{1}{2}x^2$
x^2	$\frac{1}{3}x^3$
$\sqrt{x}$	$\frac{2}{3}x^{3/2}$
x^n	$\frac{1}{n+1}x^{n+1}$ $(n \neq -1)$
$x^{-1} = \frac{1}{x}$	$\ln(x)$ $(n = -1)$
e^x	e^x
5^x	$\frac{1}{\ln(5)}5^x$
a^x	$\frac{1}{\ln(a)}a^x$ $(a>0)$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\csc^2(x)$	$-\cot(x)$
$\csc(x)\cot(x)$	$-\csc(x)$

$f(x)$	$\int f(x)\,dx - C$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$
$\frac{1}{\sqrt{1+x^2}}$	$\sinh^{-1}(x)$
$\frac{1}{\sqrt{x^2-1}}$	$\cosh^{-1}(x)$
$\frac{1}{1-x^2}$	$\tanh^{-1}(x)$

All of the basics summarized: what you need to know:

Def^{n}/ F is the antiderivative of f if $F'(x) = f(x)$. The most general anitderivative is the indefinite integral $\int f(x)\,dx$ meaning

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$$

I list below all the basic antiderivatives and they're corresponding derivatives

$\int dx = x + c$	$\frac{d}{dx}(x+c) = \frac{d}{dx}(x) + \underbrace{\frac{d}{dx}(c)}_{0} = 1$
$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$	$\frac{d}{dx}(x^n) = nx^{n-1}$
$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$	$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$
$\int \cos(x)\,dx = \sin(x) + c$	$\frac{d}{dx}(\sin(x)) = \cos(x)$
$\int \sin(x)\,dx = -\cos(x) + c$	$\frac{d}{dx}(\cos(x)) = -\sin(x)$
$\int \sec^2(x)\,dx = \tan(x) + c$	$\frac{d}{dx}(\tan(x)) = \sec^2(x)$
$\int \sec(x)\tan(x)\,dx = \sec(x) + C$	$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
$\int \csc^2(x)\,dx = -\cot(x) + c$	$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$	$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
$\int e^x dx = e^x + c$	$\frac{d}{dx}(e^x) = e^x$
$\int a^x dx = \frac{1}{\ln(a)}a^x + C$	$\frac{d}{dx}(a^x) = \ln(a)\,a^x$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$	$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$
$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$	$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

You should memorize every integral on this page. Additionally know that:

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c f(x)\,dx = c \int f(x)\,dx$$

Example 6.2.4, 6.2.5 and 6.2.6:

E4 Use linearity to break it up,

$$\int\left((x+3)^2 + \sin(x) + 3^x\right)dx = \int\left(x^2+6x+9+\sin(x)+3^x\right)dx$$

$$= \boxed{\frac{1}{3}x^3 + 3x^2 + 9x - \cos(x) + \frac{1}{\ln(3)}3^x + C}$$

E5

$$\int\left(\frac{3}{\sqrt{1-x^2}} + \frac{\pi}{1+x^2}\right)dx = 3\int\frac{1}{\sqrt{1-x^2}}dx + \pi\int\frac{1}{1+x^2}dx$$

$$= \boxed{3\sin^{-1}(x) + \pi\tan^{-1}(x) + C}$$

E6

$$\int\frac{(x+1)^3}{x}dx = \int\frac{x^3+3x^2+3x+1}{x}dx$$

$$= \int\left(x^2+3x+3+\frac{1}{x}\right)dx$$

$$= \boxed{\frac{1}{3}x^3 + \frac{3}{2}x^2 + 3x + \ln|x| + C}$$

Much more can be said about indefinite integration. All we have done so far is to pick the easiest of examples. We will learn u-substitution this semester and then a variety of other techniques to calculate indefinite integrals next semester. Generally it is a much more challenging problem than differentiation.

6.3. EXAMPLES OF DEFINITE INTEGRATION

We proved the FTC in the first section of this chapter. I now give a simple, non-rigourous proof for the FTC, maybe you will like this one more.

Fundamental Th^{m} of Calculus (FTC)

Let f be continuous on (a,b) with antiderivative F, meaning $F'(x) = f(x)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Pf:/ See page 367. The proof is not intuitively obvious, but it's basically this

$$F'(x) = f(x) \approx \frac{\Delta F}{\Delta x} \quad \text{as } \Delta x \to 0$$

$$\Rightarrow \Delta F = f(x)\Delta x$$

$$\Rightarrow \sum \Delta F = \sum f(x)\Delta x$$

$$\Rightarrow F(b) - F(a) = \int_a^b f(x)\,dx \quad //$$

***Example 6.3.1:** (we did this before, contrast with Ex. 6.1.1)*

E1 (again)

$$\int_0^3 (x+2)\,dx = \left[\tfrac{1}{2}x^2 + 2x\right]_0^3$$

$$= \left(\tfrac{1}{2}3^2 + 2(3)\right) - \left(\tfrac{1}{2}0^2 + 2(0)\right)$$

$$= \boxed{\tfrac{9}{2} + 6}$$

Example 6.3.2 and 6.3.3:

E2

$$\int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi}$$

← the bar means "evaluate $-\cos(x)$ from 0 to π"

$$= -\cos(\pi) - (-\cos(0))$$

$$= \boxed{2}$$

E3

$$\int_1^2 \frac{1}{3x}\,dx = \frac{1}{3}\int_1^2 \frac{1}{x}\,dx$$

$$= \frac{1}{3}\ln|x|\Big|_1^2$$

$$= \frac{1}{3}\ln(2) - \frac{1}{3}\ln(1)$$

$$= \boxed{\frac{1}{3}\ln(2)}$$

E4

$$\int_0^1 3^x\,dx = \frac{1}{\ln(3)}3^x\Big|_0^1 = \boxed{\frac{3}{\ln(3)}}$$

Properties of Definite Integration:

These can be shown by direct calculations from Definition 6.1.1.

We assume that a, b, c, m, M are all independent of x, they're constants.

$\int_a^b f(x)dx = -\int_b^a f(x)dx$	$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$	$\int_a^b c f(x)dx = c\int_a^b f(x)\,dx$

The next set of properties are not as often used, but are important, and are the 1st and crudest method of estimating an integral,

6.	$f(x) \geq 0$ for $a \leq x \leq b \Rightarrow \int_a^b f(x)\,dx \geq 0$
7.	$f(x) \geq g(x)$ for $a \leq x \leq b \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$
8.	$m \leq f(x) \leq M$ for $a \leq x \leq b \Rightarrow m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Example 6.3.5 and 6.3.6:

$\boxed{E5}$ $$\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx = \left.\frac{-x^2}{2}\right|_{-1}^{0} + \left.\frac{x^2}{2}\right|_{0}^{1} = \boxed{1}$$

Question: why did I break it into two pieces anyway?

$\boxed{E6}$ Notice $0 \leq \tan^{-1}(x) < \pi/2$ for $0 \leq x < \infty$ so

$$0 \leq \int_7^{11} \tan^{-1}(x)\,dx < (11-7)\frac{\pi}{2} = 2\pi$$

The problem of finding a good upper (M) or lower (m) bound for a given function f can be quite challenging. However, if the function is strictly increasing or decreasing on the interval $[a, b]$ then the values of the function endpoints of the function provide convenient bounds. Property 8. is at least gives us some estimate of the possible values that $\int_a^b f(x)dx$ may take, even if the antiderivative of $f(x)$ is out of reach.

6.4. INTEGRATION AS AN INVERSE PROCESS

Sometimes you'll hear someone say that differentiation and integration are inverse processes. This is true in a certain sense. However, unqualified this is a dangerous statement. Differentiation takes a particular function in and then it outputs another function. In contrast, indefinite integration takes in a particular function and returns a whole class of functions which differ by at most a constant. Definite integration takes in a function and returns a number which is the signed area under the curve. So, how is it that differentiation and integration are inverse processes, the inputs and outputs of the processes don't match like you'd like. As functions of functions we can say:

$$\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \qquad\qquad \int : C^\infty(\mathbb{R}) \to \mathcal{F}C^\infty(\mathbb{R})$$

The integral is an inverse to the derivative in a certain way, let's explain how. It is not quite as direct as you might like. Suppose that F is a particular antiderivative of f and $a \in dom(f)$, that is we assume $F'(x) = f(x)$. Furthermore, suppose that $x \in \mathbb{R}$ such that $[a, x] \subseteq dom(f)$, the FTC yields

$$\int_a^x f(u)du = F(x) - F(a)$$

for each such x. Now we can differentiate this with respect to x,

$$\frac{d}{dx}\int_a^x f(u)du = \frac{dF}{dx}(x) = f(x).$$

this is how differentiation and integration are "inverse processes". Now we can just as well integrate over other varying bounds, here I am thinking of A, B as being functions of x, we still find $\int_A^B f(u)du = F(B) - F(A)$ thus,

$$\begin{aligned} \frac{d}{dx}\int_A^B f(u)du &= \frac{dF}{dx}(B)\frac{dB}{dx} - \frac{dF}{dx}(A)\frac{dA}{dx} \\ &= f(B)\frac{dB}{dx} - f(A)\frac{dA}{dx}. \end{aligned}$$

We had to use the chain rule here and it produced the extra factors $\frac{dB}{dx}, \frac{dA}{dx}$. All of the examples in this section are based on these somewhat silly calculations.

Example 6.4.1:

$\boxed{E1}$ $$\frac{d}{dx}\Big(\int_3^x \underbrace{\cos(-\sqrt{u})}_{f(u)}\,du\Big) = \frac{d}{dx}\big(F(x)-F(3)\big) \quad : \text{ using FTC}$$

$$= F'(x) - 0 \quad : F(3) \text{ is a constant.}$$

$$= f(x) \quad : F'(x) = f(x).$$

$$= \boxed{\cos(\sqrt{x}).}$$

Example 6.4.2:

$\boxed{E2}$ $$\frac{d}{dx}\Big(\int_x^{x^2} e^{-t^2}dt\Big) = \frac{d}{dx}\big(F(x^2)-F(x)\big) \quad : \text{ letting } F \text{ be the antiderivative of } f(t) = e^{-t^2}.$$

$$= 2x\,F'(x^2) - F'(x) \quad : \text{ Chain Rule}$$

$$= 2x\,f(x^2) - f(x)$$

$$= \boxed{2x\,e^{-x^4} - e^{-x^2}}$$

Comment: We never needed to find F explicitly! The fact that the functions in $\boxed{E1}$ and $\boxed{E2}$ were continous gauranteed the existence of the antiderivatives.

Example 6.4.3:

$\boxed{E3}$ The sine integral function $Si(x) \equiv \int_0^x \frac{\sin(t)}{t}\,dt$ where we define $Si(0) = 0$. This function arises in Electrical Engineering and in the study of optics, (97)

$$\frac{d}{dx}\big(Si(x)\big) = \frac{d}{dx}\Big(\int_0^x \Big(\frac{\sin(t)}{t}\Big)dt\Big) \quad :$$

$$= \frac{d}{dx}\big(F(x) - F(0)\big) \quad : \text{ FTC where } f(x) = \frac{\sin(x)}{x} \text{ and } F \text{ is the antider. of } f.$$

$$= f(x) = \boxed{\frac{\sin(x)}{x}}$$

Example 6.4.4:

E4 $$\frac{d}{dx}\left(\int_{\sin(x)}^{x^3+3} \sqrt{u}\, du\right) = \frac{d}{dx}\Big(F(x^3+3) - F(\sin(x))\Big) \quad : \quad f(x) \equiv \sqrt{x}, \ F'(x) = f(x)$$

$$= 3x^2 F'(x^3+3) - \cos(x)F'(\sin(x)) \quad : \ \underline{\text{chain rule}}$$

$$= \boxed{3x^2\sqrt{x^3+3} - \cos(x)\sqrt{\sin(x)}}$$

Comment: I could have integrated $\sqrt{u}$ and then differentiated, but this is much easier.

Example 6.4.5:

E5 Let f be continuous everywhere.

$$\frac{d}{dx}\left(\int_{x^2}^{-x} f(u)\,du\right) = \frac{d}{dx}\Big(F(-x) - F(x^2)\Big) \quad : \ \text{FTC where } F'(x) = f(x)$$

$$= -F'(-x) - 2xF'(x^2)$$

$$= \boxed{-f(-x) - 2x f(x^2)}$$

I hope these examples provide enough variety for everyone to get the idea here.

6.5. INTEGRATION BY U-SUBSTITUTION

The integrations we have done up to this point have been elementary. Basically all we have used is linearity of integration and our basic knowledge of differentiation. We made educated guesses as to what the antiderivative was for a certain class of rather special functions(see pg. 143). Integration requires that you look ahead to the answer before you get there. For example, $\int \sin(x)dx$. To reason this out we think about our basic derivatives, we note that the derivative of $\cos(x)$ gives $-\sin(x)$ so we need to multiply our guess by -1 to fix it. We conclude that $\int \sin(x)dx = -\cos(x) + c$. The logic of this is essentially educated guessing. You might be a little concerned at this point. Is that all we can do? Just guess? Well, no. There is more. But, those basic guesses remain, They form the basis for all the rest of the integration we will learn for this semester and most of calculus II.

The new idea we look at in this section is called "u-substitution". It amounts to the reverse chain rule. The goal of a properly posed u-substitution is to change the given integral to a new integral which is elementary. Typically we go from an integration in x which seems incalculable to a new integration in u which is elementary. For the most part we will make direct substitutions, these have the form $u = stuff\ in\ x$ however, this is not strictly speaking the only sort of substitution that can be made. Implicitly defined substitutions such as $x = \sin(\theta)$ play a critical role in many interesting integrals, we will deal with those more subtle integrations in a later chapter.

Finally, I should emphasize that when we do a u-substitution we must be careful to convert each and every part of the integral to the new variable. This includes both the integrand($f(x)$) and the measure(dx) in an indefinite integral $\int f(x)dx$. Or the integrand($f(x)$), measure(dx) and upper and lower bounds a, b in a definite integral $\int_a^b f(x)dx$.

Example 6.5.1: (notice that we replace dx and the integrand)

E1)

$$\int x e^{x^2} dx = \int x e^u \frac{du}{2x} \longleftarrow \boxed{u = x^2 \quad \frac{du}{dx} = 2x \;\therefore\; dx = \frac{du}{2x}}$$

$$= \frac{1}{2}\int e^u du$$

$$= \frac{1}{2} e^u + C$$

$$= \boxed{\frac{1}{2} e^{x^2} + C}$$

Example 6.5.2:

E2)

$$\int (ax+b)^{13}\,dx = \int u^{13}\,\frac{du}{a}$$

$$u = ax+b$$

$$\frac{du}{dx} = a \quad \therefore \quad dx = \frac{du}{a}$$

$$= \frac{1}{14a}u^{14} + C$$

$$= \boxed{\frac{1}{14a}(ax+b)^{14} + C}$$

Example 6.5.3:

E3)

$$\int 5^{\frac{x}{3}}\,dx = \int 5^{u}(3\,du)$$

$$u = \frac{1}{3}x$$

$$\frac{du}{dx} = \frac{1}{3} \quad \therefore \quad dx = 3\,du$$

$$= 3\,\frac{5^{u}}{\ln(5)} + C$$

$$= \boxed{\frac{3}{\ln(5)}5^{x/3} + C}$$

Example 6.5.4:

E4)

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$$u = \cos(x)$$

$$\frac{du}{dx} = -\sin(x) \quad \therefore \quad dx = \frac{-du}{\sin(x)}$$

$$= \int \frac{\sin(x)}{u}\left(\frac{-du}{\sin(x)}\right)$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\cos(x)| + C} = \ln|\sec(x)| + C$$

Example 6.5.5:(the bubble mentions an implicit substitution which works here)

E5)

$$\int \frac{2x}{1+x^4}\,dx = \int \frac{2x}{1+u^2}\,\frac{du}{2x}$$

$$u = x^2$$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$= \int \frac{du}{1+u^2}$$

You can also say "you can calculate this integral using sub.

$$\int \frac{dx}{1+x^2} \quad \begin{cases} \text{let } x = \tan u \\ dx = \sec^2 u\,du \end{cases}$$

$$= \int 1\,du$$

$$= u = \tan^{-1} x$$

(looking ahead)

$$= \tan^{-1}(u) + C$$

$$= \boxed{\tan^{-1}(x^2) + C}$$

Example 6.5.6:

E6) $$\int \sqrt[3]{1-3x}\,dx = \int \sqrt[3]{u}\,\frac{du}{-3}$$

$u = 1-3x$

$\frac{du}{dx} = -3 \quad \therefore \quad dx = \frac{du}{-3}$

$$= \frac{-1}{3}\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$$

$$= -\frac{1}{3}\frac{3}{4}u^{4/3} + C$$

$$= \boxed{\frac{-1}{4}(1-3x)^{4/3} + C}$$

Example 6.5.7:

E7) $$\int \frac{1}{x+b}\,dx = \int \frac{du}{u}$$

$u = x+b$

$\frac{du}{dx} = 1 \quad \therefore \quad dx = du$

$$= \ln|u| + C$$

$$= \boxed{\ln|x+b| + C}$$

Example 6.5.8:

E8) $$\int \frac{x^2}{\sqrt{x^2-x^4}}\,dx = \int \frac{x^2}{x\sqrt{1-x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{u}}\,\frac{du}{-2x}$$

$u = 1-x^2$

$\frac{du}{dx} = -2x \quad \therefore \quad dx = \frac{du}{-2x}$

$$= \frac{-1}{2}\int u^{-1/2}\,du$$

$$= -\frac{1}{2}\,2u^{1/2} + C$$

$$= \boxed{-\sqrt{1-x^2} + C}$$

Example 6.5.9:

E9) $$\int \frac{\ln(x)}{x}\,dx = \int \frac{u}{x}\,x\,du$$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x} \quad \therefore \quad dx = x\,du$

$$= \int u\,du$$

$$= \frac{1}{2}u^2 + C$$

$$= \boxed{\frac{1}{2}(\ln(x))^2 + C}$$

Example 6.5.10:

E10

$$\int \sin(3\theta)\,d\theta = \int \sin(u)\,\frac{du}{3}$$

$$= \frac{-1}{3}\cos(u) + C$$

$$= \boxed{\frac{-1}{3}\cos(3\theta) + C}$$

$$\boxed{\begin{array}{l} u = 3\theta \\ \frac{du}{d\theta} = 3 \quad \therefore \quad d\theta = \frac{du}{3} \end{array}}$$

Example 6.5.11:

E11

$$\int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx = \int \frac{u}{\sqrt{1-x^2}}\left(\sqrt{1-x^2}\,du\right)$$

$$= \int u\,du$$

$$= \frac{1}{2}u^2 + C$$

$$= \boxed{\frac{1}{2}\left(\sin^{-1}(x)\right)^2 + C}$$

$u = \sin^{-1}(x)$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \therefore \quad dx = \sqrt{1-x^2}\,du$

Remark: $x = \sin(u)$ will work.

Example 6.5.12:

E12

$$\int t\cos(t^2+\pi)\,dt = \int t\cos(u)\,\frac{du}{2t}$$

$$= \frac{1}{2}\int \cos(u)\,du$$

$$= \frac{1}{2}\sin(u) + C$$

$$= \boxed{\frac{1}{2}\sin(t^2+\pi) + C}$$

$$\boxed{\begin{array}{l} u = t^2 + \pi \\ \frac{du}{dt} = 2t \quad \therefore \quad dt = \frac{du}{2t} \end{array}}$$

Example 6.5.13:

E13

$$\int \sin^3\theta\,d\theta = \int (1-\cos^2\theta)\sin\theta\,d\theta$$

$$= \int (1-u^2)\sin\theta\,\frac{du}{-\sin\theta}$$

$$= \int (u^2-1)\,du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \boxed{\frac{1}{3}\cos^3\theta - \cos\theta + C}$$

$$\boxed{\begin{array}{l} u = \cos\theta \\ \frac{du}{d\theta} = -\sin\theta \quad \therefore \quad d\theta = \frac{du}{-\sin\theta} \end{array}}$$

Example 6.5.14:

E14)

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1/a^2}{a^2/a^2 + x^2/a^2}\,dx \quad , \text{ assume } a \neq 0$$

$$= \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx$$

$$= \frac{1}{a^2}\int \frac{1}{1+u^2}\,a\,du \longleftarrow \boxed{u = x/a,\quad \frac{du}{dx} = \frac{1}{a} \Rightarrow dx = a\,du}$$

$$= \frac{1}{a}\tan^{-1}(u) + c$$

$$= \boxed{\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c}$$

Definite Integrals involving u-subtitution:

There are two ways to do these. It is best if you understand both methods.

i. Find the antiderivative via u-substitution and then use the FTC to evaluate in terms of the given upper and lower bounds in x. (see E15 below)

ii. Do the u-substitution and change the bounds all at once, this means you will use the FTC and evaluate the upper and lower bounds in u. (see E16 below)

I will deduct points if you write things like a definite integral is equal to an indefinite integral (just leave off the bounds during the u-substitution). The notation is not decorative, it is necessary and important to use correct notation.

Example 6.5.15:(we use Example 6.5.12 to get started)

E15)

$$\int_0^{\sqrt{\pi/2}} t\cos(t^2+\pi)\,dt = \left[\frac{1}{2}\sin(t^2+\pi) + c\right]_0^{\sqrt{\pi/2}}$$

$$= \left[\frac{1}{2}\sin\left(\frac{\pi}{2}+\pi\right) + \cancel{c}\right] - \left[\frac{1}{2}\sin(\pi) + \cancel{c}\right]$$

$$= \frac{1}{2}\sin(3\pi/2)$$

$$= \boxed{-1/2}$$

This illustrates method i.) we find the antiderivative off to the side then calculate the integral using the FTC in the x-variable. Well, the t-variable here. This is a two-step process. In the next example I'll work the same integral using method ii.). In constrast, that is a one-step process but the extra step is that you need to change the bounds in that scheme. Generally, some problems are easier with both methods. Also, sometimes you may be faced with an abstract question which demands you understand method ii.). In short, you should strive to understand both methods.

Example 6.5.16:(same as Example 6.5.15 but with method ii.)

E16)
$$\int_0^{\sqrt{\pi/2}} t\cos(t^2+\pi)\,dt = \int_{\pi}^{3\pi/2} t\cos(u)\frac{du}{2t} \leftarrow$$

$u = u(t) = t^2+\pi$
$\frac{du}{dt} = 2t \therefore dt = \frac{du}{2t}$
$u(0) = \pi$
$u(\sqrt{\pi/2}) = \pi/2 + \pi = \frac{3\pi}{2}$

$$= \frac{1}{2}\int_{\pi}^{3\pi/2}\cos(u)\,du$$
$$= \frac{1}{2}\sin(u)\Big|_{\pi}^{3\pi/2}$$
$$= \frac{1}{2}\sin(3\pi/2) - \frac{1}{2}\sin(\pi)$$
$$= \boxed{\frac{-1}{2}}$$

The bounds must change when we change the variable of integration.

Of course both methods give the same result. This is good, otherwise math would be much harder. Or perhaps it would be easier, but useless. I digress.

Example 6.5.17:

E17)
$$\int_{4\pi^2}^{9\pi^2}\frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int_{2\pi}^{3\pi}\frac{\sin(u)}{\sqrt{x}}2\sqrt{x}\,du \leftarrow$$

$u = u(x) = \sqrt{x}$
$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \therefore dx = 2\sqrt{x}\,dx$
$u(4) = \sqrt{4\pi^2} = 2\pi$
$u(9) = \sqrt{9\pi^2} = 3\pi$

$$= \int_{2\pi}^{3\pi} 2\sin(u)\,du$$
$$= -2\cos(u)\Big|_{2\pi}^{3\pi}$$
$$= -2\cos(3\pi) + 2\cos(2\pi)$$
$$= \boxed{4}$$

Example 6.5.18:

E18] $\displaystyle\int_0^{\pi/4} \tan^3\theta\, d\theta = \left[\ln|\cos\theta| + \frac{1}{2\cos^2\theta}\right]_0^{\pi/4} = \left(\ln\left|\tfrac{\sqrt{2}}{2}\right| + 1\right) - \left(\ln(1) + \tfrac{1}{2}\right)$ using work below.

$= \boxed{\ln(\sqrt{2}/2) + \tfrac{1}{2}}$

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$$\int \tan^3\theta\, d\theta = \int \frac{\sin^3\theta}{\cos^3\theta}\, d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos^3\theta} \sin\theta\, d\theta$$

$$= \int \frac{1-u^2}{u^3}(-du)$$

$u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta \Rightarrow -du = \sin\theta\, d\theta$

$$= \int \left(\frac{1}{u} - \frac{1}{u^3}\right) du$$

$$= \ln|u| + \frac{1}{2u^2} + C$$

$$= \ln|\cos\theta| + \frac{1}{2\cos^2\theta} + C$$

Example 6.5.19:

E19] $\tan^2\theta + 1 = \sec^2\theta$ and $\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$

$$\int \sec^6\theta\, d\theta = \int \sec^4\theta \sec^2\theta\, d\theta$$

$$= \int (1+\tan^2\theta)^2 \sec^2\theta\, d\theta$$

$$= \int (1+u^2)^2\, du$$

$u = \tan\theta$

$\frac{du}{d\theta} = \sec^2\theta \Rightarrow \sec^2\theta\, d\theta = du$

$$= \int (1 + 2u^2 + u^4)\, du$$

$$= u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \boxed{\tan\theta + \frac{2}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + C}$$

So we can then calculate definite integrals using the above result,

$$\int_0^{\pi/4} \sec^6\theta\, d\theta = \left[\tan\theta + \frac{2}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta\right]_0^{\pi/4}$$

$$= \left(\tan(\pi/4) + \frac{2}{3}\tan^3(\pi/4) + \frac{1}{5}\tan^5(\pi/4)\right) - 0$$

since $\tan(0) = 0$ and $\tan(\pi/4) = 1$

$$= 1 + \frac{2}{3} + \frac{1}{5}$$

$$= \frac{15 + 10 + 3}{15}$$

$$= \boxed{\frac{28}{15}}$$

Example 6.5.20:(this substitution is at a whole other level of insight)

$\boxed{E20}$ The easiest sol$^{\underline{n}}$ is often the most clever one, (104)

$$\int \sec(\theta)\,d\theta = \int \frac{du}{u} \longleftarrow \text{see below}^{*} \text{ for why} \quad \boxed{\begin{array}{l} u = \sec\theta + \tan\theta \\ \dfrac{du}{u} = \sec\theta\,d\theta \end{array}}$$

$$= \ln|u| + c$$

$$= \boxed{\ln|\sec\theta + \tan\theta| + C}$$

$*$ Where the substitution followed from

$$\frac{d}{d\theta}\left(\sec\theta + \tan\theta\right) = \sec\theta\tan\theta + \sec^2\theta = \sec\theta(\tan\theta + \sec\theta)$$

In other words since $u = \sec\theta + \tan\theta$,

$$\frac{du}{d\theta} = (\sec\theta)u \implies \frac{du}{u} = \sec\theta\,d\theta \quad \left(\begin{array}{l}\text{as we} \\ \text{claimed above}\end{array}\right)^{*}.$$

- There are other ways to do this integral, but this is by far the most efficient sol$^{\underline{n}}$: See below the less inspired method,

You might ask the question why choose the u-substitution given in E20? The answer is experience, calculation and a lot of creativity. There is a less inspired way to do this integral, but the method below uses the technique of partial fractions. We will treat that technique next semester, I include here just to accompany E20.

$$\int \sec\theta\,d\theta = \int \frac{1}{\cos\theta}\,d\theta$$

$$= \int \frac{\cos\theta\,d\theta}{\cos^2\theta}$$

$$= \int \frac{\cos\theta\,d\theta}{1-\sin^2\theta}$$

$$= \int \frac{1}{1-u^2}\,du \longleftarrow \boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}}$$

$$= \frac{1}{2}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right) du \longleftarrow \boxed{\text{Partial Fractal Decomposition.}}$$

$$= \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right) + C$$

$$= \boxed{\frac{1}{2}\left(\ln|1+\sin\theta| - \ln|1-\sin\theta|\right) + C}$$

Clearly the U-substitution is easier, if you know to pick it.

$\boxed{\text{\underline{Bonus Point}: Show that the two answers are the same despite their apparent difference}}$

(Hand-in to me soon please.)

(***The calculation above is not part of the required material for calculus I)***

Summary of U-substitution:

We look at the given problem and try to see if there is some way to reduce it to an elementary integral. There is no general method to choose u. Frankly, it requires creativity. If you want to know the algorithm then the best answer I can give you is to work many problems, then work some more. Have you worked enough? No. Go work some more. Once it is completely boring and you can see the u-substitution as soon as you see the integral then your done, you're ready. Before then, you should worry. But, turn that frown upside down with diligent work, don't despair, just get back to the examples and leave no stone unturned. Be warned, I expect mastery of this topic. I will ask challenging questions.