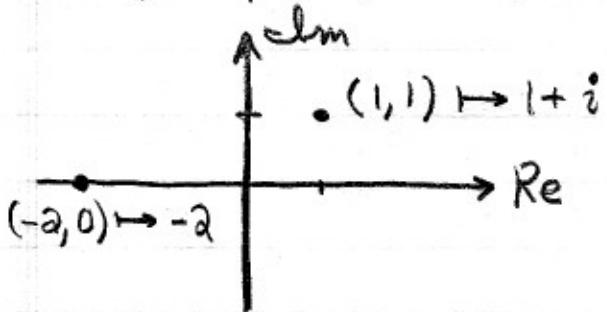


COMPLEX VARIABLES : A SHORT INTRODUCTION

A complex variable is a variable whose values reside in the complex numbers. We denote the complex numbers by \mathbb{C} . If z is a complex number then we say $z \in \mathbb{C}$. Every complex number has a real & imaginary part, $a, b \in \mathbb{R}$

$$z = a + ib \Rightarrow \operatorname{Re}\{z\} = a \text{ & } \operatorname{Im}\{z\} = b$$

This shows how $z \in \mathbb{C}$ can be identified as a point in a plane; $z \mapsto (\operatorname{Re}\{z\}, \operatorname{Im}\{z\}) = (a, b)$. This is the complex plane, it represents complex numbers.



Remark: a complex # is also a 2-dimensional vector.

Usually an eqⁿ involving complex variables has two real variable eqⁿ's hidden within it. For example suppose we have the complex eqⁿ, $z^2 = z$ where $z = x + iy$, here $\operatorname{Re}\{z\} = x$ & $\operatorname{Im}\{z\} = y$. Both x & y are real variables. Consider then,

$$\begin{aligned} z^2 = z &\Rightarrow (x+iy)^2 = x+iy \\ &\Rightarrow x^2 + 2ixy + i^2 y^2 = x+iy \\ &\Rightarrow x^2 - y^2 + i(2xy) = x+iy \leftarrow (*) \\ &\Rightarrow \underbrace{x^2 - y^2}_{\substack{\text{Real Part of} \\ \text{the Eq}^n (*)}} = x \quad \& \quad \underbrace{2xy}_{\substack{\text{Imaginary Part of} \\ \text{the Eq}^n (*)}} = y \end{aligned}$$

I just used $i^2 = -1$, next lets collect all the basic rules for complex arithmetic & algebra,

PROPERTIES AND DEFINITIONS

Suppose $Z = x+iy$ and $W = a+ib$ where x, y, a, b are real variables,

$$\textcircled{1} \quad ZW = (x+iy)(a+ib) \equiv (xa - yb) + i(xb + ya)$$

$$\textcircled{2} \quad Z^* \equiv x - iy$$

$$\textcircled{3} \quad ZW = WZ$$

$$\textcircled{4} \quad \text{If } Z \neq 0 \text{ then } \frac{1}{Z} = \frac{x-iy}{x^2+y^2}.$$

\equiv means its definition

$$\textcircled{5} \quad \frac{1}{i} = -i$$

$$\textcircled{6} \quad Z^*Z = x^2 + y^2$$

Let me prove $\textcircled{4}$, we need $Z\left(\frac{1}{Z}\right) = 1$.

$$Z\left(\frac{1}{Z}\right) = (x+iy)\left(\frac{x-iy}{x^2+y^2}\right) = \frac{x^2 - ix^2y + ix^2y - i^2y^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1.$$

we need $Z \neq 0$ to insure that $x^2+y^2 \neq 0$.

Example: find $\frac{1}{1+i}$. Basically we just follow $\textcircled{4}$,

$$\frac{1}{1+i} = \frac{1-i}{1^2+1^2} = \frac{1-i}{2}. \text{ In other words, } (1+i)^{-1} = \frac{1}{2} - \frac{i}{2}$$

$$\begin{aligned} \text{Now lets prove } \textcircled{6}, \quad Z^*Z &= (x-iy)(x+iy) \\ &= x^2 + ix^2y - ixy^2 - i^2y^2 \\ &= x^2 + y^2. \end{aligned}$$

Notice we could also write the reciprocal in terms of Z, Z^* ,

$$\frac{1}{Z} = \frac{Z^*}{Z^*Z}$$

the operation "*" is called complex conjugation it has many nice properties, $(Z+W)^* = Z^* + W^*$ and $(ZW)^* = Z^*W^*$.

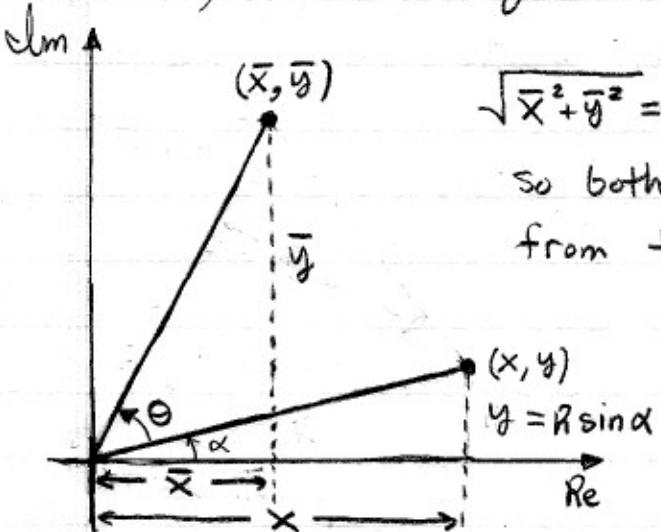
Up to now these ideas should be review from high school.
What follows is more useful and is likely new to you.

Euler's Identity: $e^{i\theta} = \cos\theta + i\sin\theta$

Let me attempt to elucidate the geometric foundations of this expression. Let's consider a point $z = x+iy$, for graphical convenience take $x, y > 0$. Consider,

$$\begin{aligned} e^{i\theta} z &= (\cos\theta + i\sin\theta)(x+iy) \\ &= \cos\theta x - \sin\theta y + i(\sin\theta x + \cos\theta y) \\ &= \bar{x} + i\bar{y} \end{aligned}$$

where I've defined $\bar{x} = x\cos\theta - y\sin\theta$ & $\bar{y} = x\sin\theta + y\cos\theta$
if you had studied rotations in the plane before these would be familiar, but in case you haven't let's draw the picture



$\sqrt{\bar{x}^2 + \bar{y}^2} = \sqrt{(\cos\theta x - \sin\theta y)^2 + (\sin\theta x + \cos\theta y)^2} = \sqrt{x^2 + y^2}$
so both (x, y) & (\bar{x}, \bar{y}) are distance $R = \sqrt{x^2 + y^2}$ from the origin. I let " α " be the standard angle relative to the Re axis in the CCW direction.
it's easy to see that $x = R\sin\alpha$ & $y = R\cos\alpha$

whereas clearly (\bar{x}, \bar{y}) is at $\theta + \alpha$ so $\bar{x} = R\sin(\alpha + \theta)$ & $\bar{y} = R\cos(\alpha + \theta)$.

$$\bar{x} = R\sin(\alpha + \theta) = R\sin\alpha\cos\theta + R\cos\alpha\sin\theta = y\cos\theta + x\sin\theta$$

$$\bar{y} = R\cos(\alpha + \theta) = R\cos\alpha\cos\theta - R\sin\alpha\sin\theta = x\cos\theta - y\sin\theta$$

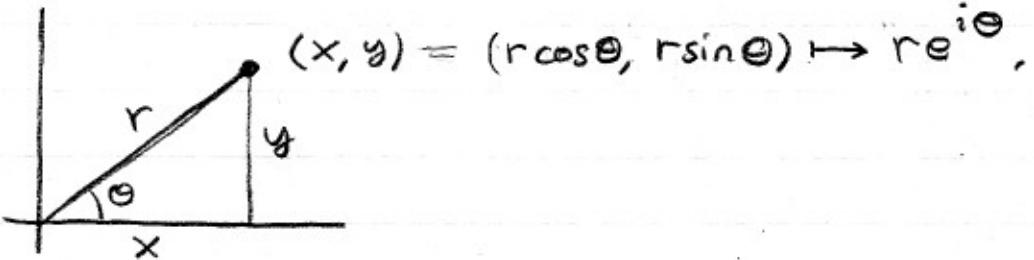
adding angles formulas for sin & cos.

Thus multiplying by $e^{i\theta} = \cos\theta + i\sin\theta$ rotates the point by θ .

(4i)

POLAR FORM OF COMPLEX NUMBER

Given $z = x+iy$ we can use $e^{i\theta} = \cos\theta + i\sin\theta$ to rewrite z as $z = re^{i\theta}$ where $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$.



so I used the adding angles trig. identities to help this identification make sense. However, we can take another perspective, assume $e^{i\theta} = \cos\theta + i\sin\theta$ then derive all sorts of identities from this simple fact. Well we also assume a few other properties to start,

PROPERTIES OF exp of COMPLEX NUMBER

Let $z = x+iy$ and $w = a+ib$ then

- ① $e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y$
- ② $e^{z+w} = e^z e^w$
- ③ for $n \in \mathbb{R}$ $(e^z)^n = e^{nz}$

with these few simple rules we'll be able to derive just about any trig. identity we could possibly need.

Notice: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

Adding & subtracting yields two formulas worth remembering,

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

DERIVING TRIGONOMETRIC IDENTITIES

I'll proceed by example. Keep the previous page in mind,

Example:

$$\begin{aligned}\cos^2 \theta + \sin^2 \theta &= \left[\frac{1}{2}(e^{i\theta} + e^{-i\theta}) \right]^2 + \left[\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \right]^2 \\&= \frac{1}{4}(e^{2i\theta} + e^{i\theta}e^{-i\theta} + e^{-i\theta}e^{i\theta} + e^{-2i\theta}) \\&\quad - \frac{1}{4}(e^{2i\theta} - e^{i\theta}e^{-i\theta} - e^{-i\theta}e^{i\theta} + e^{-2i\theta}) \quad \text{note } e^{i\theta}e^{-i\theta} = e^0 = 1 \\&= \frac{1}{4}(1+1) - \frac{1}{4}(-1-1) = \frac{4}{4} = 1.\end{aligned}$$

Now this not that surprising, hopefully you already knew this one.
How about $\sin(2\theta) = 2\sin\theta\cos\theta$?

Example:

$$\begin{aligned}2\sin\theta\cos\theta &= 2 \cdot \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \cdot \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\&= \frac{1}{2i}(e^{2i\theta} + 1 - 1 - e^{-2i\theta}) \\&= \frac{1}{2i}(e^{2i\theta} - e^{-2i\theta}) = \sin(2\theta).\end{aligned}$$

Do you know another way to derive this fact? How about?

Example:

$$\begin{aligned}\cos^2 \theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\&= \frac{1}{4}(e^{2i\theta} + 2 + e^{-2i\theta}) \\&= \frac{1}{2}(1 + \frac{1}{2}(e^{2i\theta} + e^{-2i\theta})) \\&= \frac{1}{2}(1 + \cos(2\theta)).\end{aligned}$$

(6i)

Example:

$$\begin{aligned}
 \sin(\theta)\cos(2\theta) &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) \\
 &= \frac{1}{4i}(e^{3i\theta} + e^{-i\theta} - e^{i\theta} - e^{-3i\theta}) \\
 &= \frac{1}{2}\frac{1}{2i}(e^{3i\theta} - e^{-3i\theta}) - \frac{1}{2}\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \\
 &= \frac{1}{2}\sin(3\theta) - \frac{1}{2}\sin(\theta).
 \end{aligned}$$

Example:

$$\begin{aligned}
 \sin^2\theta \cos^2\theta &= \left[\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \right]^2 \left[\frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) \right]^2 \\
 &= \frac{-1}{16} [e^{2i\theta} - 2 + e^{-2i\theta}] [e^{2i\theta} + 2 + e^{-2i\theta}] \\
 &= \frac{-1}{16} [e^{4i\theta} + 2e^{2i\theta} + 1 \\
 &\quad - 2e^{3i\theta} - 4 - 2e^{-2i\theta} \\
 &\quad + 1 + 2e^{-2i\theta} + e^{-4i\theta}] \\
 &= \frac{-1}{16} [e^{4i\theta} + e^{-4i\theta}] + \frac{1}{8} \\
 &= -\frac{1}{8} \cos(4\theta) + \frac{1}{8}.
 \end{aligned}$$

Now there are often ways of combining old trig. identities to get new ones, my point here is you can also choose a less clever route & just grind em' out via the imaginary exponentials. A more clever sol² would be,

$$\begin{aligned}
 \sin^2\theta \cos^2\theta &= (1 - \cos^2\theta) \cos^2\theta & \rightarrow &= \frac{1}{2}(1 + \cos 2\theta) - \frac{1}{4}(1 + 2\cos 2\theta) \\
 &= \cos^2\theta - \cos^4\theta & & - \frac{1}{4}\cos^2(2\theta) \\
 &= \frac{1}{2}(1 + \cos 2\theta) - \frac{1}{4}(1 + \cos(2\theta))^2 & = & \frac{1}{4} - \frac{1}{8}(1 - \cos(4\theta)) = \frac{1}{8} - \frac{1}{8}\cos 4\theta
 \end{aligned}$$

ADDING ANGLES FORMULAS

These follow from an indirect argument (the direct argument is harder),

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b)$$

On the other hand we can use laws of exponents,

$$\begin{aligned} e^{i(a+b)} &= e^{ia} e^{ib} \\ &= (\cos a + i\sin a)(\cos b + i\sin b) \\ &= (\cos a \cos b - \sin a \sin b) + i(\cos a \sin b + \sin a \cos b) \end{aligned}$$

Therefore we find that,

$$\cos(a+b) + i\sin(a+b) = (\cos a \cos b - \sin a \sin b) + i(\cos a \sin b + \sin a \cos b)$$

Then we can read two real eq's from this,

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

Given these we can derive the formula for $\tan(a+b)$,

$$\begin{aligned} \tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} = \frac{(\cos(a)\sin(b) + \sin(a)\cos(b))}{(\cos(a)\cos(b) - \sin(a)\sin(b))} \cdot \frac{\frac{1}{\cos a \cos b}}{\frac{1}{\cos a \cos b}} \\ &= \frac{\tan(b) + \tan(a)}{1 - \tan(a)\tan(b)} = \tan(a+b) \end{aligned}$$

De Moivre's THEOREM

We take Euler's Identity $e^{i\theta} = \cos\theta + i\sin\theta$ and raise it to the n^{th} power, notice $(e^{i\theta})^n = e^{ni\theta}$ so

$$e^{ni\theta} = \cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

this is the theorem, it has many hidden treasures.

$$\begin{aligned} n=2 \quad \cos(2\theta) + i\sin(2\theta) &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta \end{aligned}$$

Equating Re and Im parts we find two nice facts,

$$\boxed{\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta \end{aligned}}$$

$$\begin{aligned} n=3 \quad \cos(3\theta) + i\sin(3\theta) &= (\cos\theta + i\sin\theta)^3 \\ &= (\cos\theta + i\sin\theta)(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta) \\ &= \cos^3\theta - \sin^2\theta\cos\theta + 2i\sin\theta\cos^2\theta \\ &\quad + i\sin\theta\cos^2\theta - i\sin^3\theta - 2\sin^2\theta\cos\theta \\ &= \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\sin\theta\cos^2\theta - \sin^3\theta) \end{aligned}$$

Equating Re & Im parts reveals another pair of identities,

$$\boxed{\begin{aligned} \cos(3\theta) &= \cos^3\theta - 3\sin^2\theta\cos\theta \\ \sin(3\theta) &= 3\sin\theta\cos^2\theta - \sin^3\theta \end{aligned}}$$

Remark: I'm not a big fan of this Thⁿ, I think the direct approach yields identities of interest quicker. Anyway there are many other tricks but we'll content ourselves with those discussed thus far.

Remark: There is much more to say, I have bits & pieces scattered throughout the notes. Also I recommend the classic text by Churchill.