Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 1 Your signature below indicates you have:

(a.) I read Chapter 1 and 2 of Cook's lecture notes: ____________________.

Problem 2 Let $S = \{a, b, c\}$ where a, b, c are **distinct** elements of S.

(a.) if we view S as a **set** then $S \cup \{a, b\}$ is:

(b.) if we view S as a **multiset** then $S \cup \{a, b\}$ is:

Problem 3 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$. Let $a \in \mathbb{R}$. Find the inverse image of the singleton $\{a\}$. Break into cases as needed.

Problem 4 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x-1}{x-3}$ for all $x \in \mathbb{R} - \{3\}$. Also, define $f(3) = 1$. Show that f is a 1-1 and onto function. In other words, show that f is both injective and surjective or, simply, show f is a bijection.

Problem 5 Let $a_i, b_i \in \mathbb{C}$ for all $i \in \mathbb{N}$. Prove by induction that $\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n}(a_i + b_i)$. Your proof should include explicit comment about the use of definition of the finite sum as well as the properties of complex arithmetic. (we can replace $\mathbb{C}$ with any other set which allows addition just the same, I chose $\mathbb{C}$ to be concrete.)

Problem 6 A matrix is defined by $A_{ij} = i + j^2$ for $1 \leq i \leq 2$ and $1 \leq j \leq 3$. Write A explicitly.

Problem 7 Write the following system of equations in matrix-column notation $Av = b$ (indicate A,v and b explicitly)

$$x + y + z = 18, \qquad 2x - z = -3, \qquad x + y - z = 0.$$

Problem 8 For the previous problem, write the system as an augmented coefficient matrix. Then find solution **set** via row-reduction. Explicitly denote each row-operation with the short-hand arrow notation as shown in lecture.

Problem 9 Write the following system of equations in matrix-column notation $Av = b$ (indicate A,v and b explicitly)

$$x_1 + x_2 + x_3 + x_4 = 10, \qquad 2x_1 + 2x_2 + 4x_4 = 4$$

Problem 10 For the previous problem, write the system as an augmented coefficient matrix. Then find the standard form of the solution **set** via row-reduction. Explicitly denote each row-operation with the short-hand arrow notation as shown in lecture.

Problem 11 Write the following system of equations in matrix-column notation $Av = b$ (indicate A,v and b explicitly)

$$x + y + z = 3, \qquad 2x + y + z = 5, \qquad 3x - y + z = 0, \qquad 2x + 2y + 2z = 7.$$

Problem 12 For the previous problem, write the system as an augmented coefficient matrix. Then find the standard form of the solution **set** via row-reduction. Explicitly denote each row-operation with the short-hand arrow notation as shown in lecture.

Problem 13 Indicate with $*$-notation for unknown, possibly nonzero numbers, all the possible formats for the **reduced row echelon form** of a 2×4 matrix. For example, for a 1×2 matrix A the possibile formats of rref(A) are $[0\ 0]$, $[1\ *]$ and $[0\ 1]$.

Problem 14 Find all solutions of the matrix equation $\begin{bmatrix} x^2 & y \\ y^2 & x \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 9 & 2 \end{bmatrix}$.

Problem 15 Let E_{ij} be 2×2 matrices defined by $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ for $i, j, k, l \in \mathbb{N}_2 = \{1, 2\}$. Write the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as linear combination of the standard unit-matrices E_{ij}.

Problem 16 Find all quadratic polynomials whose graphs contain $(1, 1), (2, 1), (3, 1)$.

Problem 17 Find all polynomials whose graphs contain $(1, 1), (2, 1), (3, 1)$.

Problem 18 Solve the system

$$\sin\alpha + \cos\beta + 2\sin\gamma = \sqrt{2} + 1, \qquad 2\sin\alpha + 3\cos\beta - \sin\gamma = 1 - \sqrt{2}, \qquad 3\sin\alpha + \cos\beta = 3$$

by making a change of variables. How many solutions do you obtain? Why does this not contradict the general comment we made about the number of solutions to a linear system?

Problem 19 Consider $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. List all the distinct 2×2 matrices over $\mathbb{Z}/2\mathbb{Z}$.

Problem 20 Consider $\mathbb{Z}/6\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \ldots \bar{5}\}$. Scalar multiply $\begin{bmatrix} \bar{0} & \bar{1} & \bar{2} \\ \bar{3} & \bar{4} & \bar{5} \end{bmatrix}$ by $\bar{3}$. What is strange about the products in the $(1, 3)$ and $(2, 2)$ components?

MISSION 1 SOLUTION:

PROBLEM 2 Let $S = \{a, b, c\}$ where a, b, c are distinct elements of S

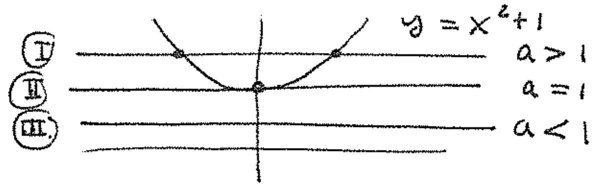
(a.) As a set $S \cup \{a, b\} = \{a, b, c\}$

(b.) As a multiset the union gives $S \cup \{a, b\} = \{a, b, c, a, b\}$
(other orders are fine, multisets need not be ordered)

Remark: my point in assigning this is simply to draw your attention to the fact we use multisets mostly. Thus, when I say "set" our custom is to mean "multiset".

PROBLEM 3 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$. Find $f^{-1}\{a\}$ for $a \in \mathbb{R}$.

A graph helps guide my logic



(I.) If $a > 1$ then $f^{-1}\{a\} = \{x \in \mathbb{R} \mid f(x) = x^2 + 1 = a\}$

Note $a > 1 \Rightarrow a - 1 > 0$ hence $x^2 + 1 = a \Rightarrow x^2 = a - 1 > 1$ has two sol^{ns}; $x = \pm\sqrt{a-1}$. That is, $\boxed{f^{-1}\{a\} = \{\pm\sqrt{a-1}\}}$ $(a > 1)$

(II.) If $a = 1$ then $f^{-1}\{a\} = \{x \in \mathbb{R} \mid x^2 + 1 = 1\}$

$= \{x \in \mathbb{R} \mid x^2 = 0\}$

$\Rightarrow \boxed{f^{-1}\{a\} = \{0\} = f^{-1}\{1\}}$ $(a = 1)$

(III.) If $a < 1$ then $x^2 + 1 = a \Rightarrow x^2 = a - 1$ and $a < 1 \Rightarrow a - 1 < 0$ hence $x^2 = a - 1$ has no sol^{n} as $x^2 \geq 0$ and we conclude

$\boxed{f^{-1}\{a\} = \phi}$ $(a < 1)$

PROBLEM 4

$$f(x) = \begin{cases} 1 & \text{if } x = 3 \\ \frac{x-1}{x-3} & \text{if } x \neq 3 \end{cases}$$

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man behind curtain:

$$y = \frac{x-1}{x-3} \longrightarrow yx - 3y = x - 1$$
$$\longrightarrow yx - x = 3y - 1$$
$$\longrightarrow x(y-1) = 3y - 1$$
$$\longrightarrow x = \frac{3y-1}{y-1} = f^{-1}(y) \text{ for } y \neq 1.$$

To find x such that $f(x) = y$ simply note, for $y \neq 1$, $f^{-1}(f(x)) = f^{-1}(y) \longrightarrow x = f^{-1}(y)$. To show 1-1 if $f(x) = f(y) \Rightarrow f^{-1}(f(x)) = f^{-1}(f(y)) \Rightarrow x = y$ (only works for all $x, y \in \mathbb{R}$ if we also define $f^{-1}(1) = 3$.

Proof: Let $y \in \mathbb{R}$. If $y = 1$ then note $f(3) = 1$. Otherwise, observe

$$f\left(\frac{3y-1}{y-1}\right) \overset{*}{=} \frac{\frac{3y-1}{y-1} - 1}{\frac{3y-1}{y-1} - 3} = \frac{3y-1-(y-1)}{3y-1-3(y-1)} = \frac{2y}{2} = y.$$

However, to be careful, we should check $\frac{3y-1}{y-1} \neq 3$ as the calculation $*$ is only valid if the inequality holds, consider,

$$\frac{3y-1}{y-1} = 3 \Rightarrow \cancel{3y} - 1 = \cancel{3y} - 3 \Rightarrow -1 = -3 \therefore \text{no sol}^{\underline{n}} \text{ exists.}$$

Hence we have shown f is a surjection.

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Suppose $f(a) = f(b)$ for some $a, b \in \mathbb{R}$. If $a = b = 3$ then $1 = 1$. Ok, more to the point, suppose $a = 3$ then $1 = f(b)$ implies either $1 = 1$ in case $b = 3$, or $b \neq 3$ and $1 = \frac{b-1}{b-3} \Rightarrow b - 3 = b - 1$ hence no such b exists. Next, suppose $a \neq 3$ then $\frac{a-1}{a-3} = \frac{b-1}{b-3}$ or $\frac{a-1}{a-3} = 1$ (no sol$^{\underline{n}}$ as we just argued). Consider, $\frac{a-1}{a-3} = \frac{b-1}{b-3} \Rightarrow ab - b - 3a + 3 = ab - a - 3b + 3$

Hence, $2a = 2b \Rightarrow a = b \Rightarrow f$ is 1-1.

PROBLEM 5 Let $a_i, b_i \in \mathbb{C}$ $\forall i \in \mathbb{N}$. Show (Claim:

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \quad \forall n \in \mathbb{N})$$

We proceed by induction on n. Observe $n=1$ gives

$$\sum_{i=1}^{1} (a_i + b_i) = a_1 + b_1 = \sum_{i=1}^{1} a_i + \sum_{i=1}^{1} b_i$$ by def$^{\underline{n}}$ of finite sum. ~~Fa~~ Thus the claim is true for $n=1$.

Suppose the claim is true for $n = k \geq 1$ and consider,

$$\begin{aligned}
\sum_{i=1}^{k+1} (a_i + b_i) &= a_{k+1} + b_{k+1} + \sum_{i=1}^{k} (a_i + b_i) && : \text{def}^{\underline{n}} \text{ of } \textstyle\sum_{i=1}^{k} \\
&= a_{k+1} + b_{k+1} + \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i && : \text{induction hypothesis.} \\
&= \left(a_{k+1} + \sum_{i=1}^{k} a_i\right) + \left(b_{k+1} + \sum_{i=1}^{k} b_i\right) && : \text{commutative property of complex addition} \\
&= \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i && : \text{def}^{\underline{n}} \text{ of } \textstyle\sum
\end{aligned}$$

Thus $k \Rightarrow k+1$ and we conclude by Proof by Mathematical Induction the claim is true for all $n \in \mathbb{N}$ //

PROBLEM 6 $A_{ij} = i + j^2$ for $1 \leq i \leq 2$, $1 \leq j \leq 3$

$$A = \left[\begin{array}{c|c|c} 1+1^2 & 1+2^2 & 1+3^2 \\ \hline 2+1^2 & 2+2^2 & 2+3^2 \end{array}\right] = \boxed{\begin{bmatrix} 2 & 5 & 10 \\ 3 & 6 & 11 \end{bmatrix}} //$$

PROBLEM 7 OOPS! I should warn you, this is out of place!

$$\begin{aligned} x + y + z &= 18 \\ 2x - z &= -3 \\ x + y - z &= 0 \end{aligned} \longrightarrow \boxed{\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 18 \\ -3 \\ 0 \end{bmatrix}}_{b}}$$

See Chapter 3 page ≈ 51 for 1st appearance... Well, more to the point, I should include this in Chapter 2 in my next version ☺

PROBLEM 8

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 18 \\ 2 & 0 & -1 & -3 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[r_3 - r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 18 \\ 0 & -2 & -3 & -39 \\ 0 & 0 & -2 & -18 \end{array}\right] \xrightarrow{r_3/-2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 18 \\ 0 & -2 & -3 & -39 \\ 0 & 0 & +1 & 9 \end{array}\right] \xrightarrow[r_2 + 3r_3]{r_1 - r_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & -2 & 0 & -12 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

$$\xrightarrow{r_2/-2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 9 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 9 \end{array}\right] \quad \therefore \quad \begin{aligned} x &= 3 \\ y &= 6 \\ z &= 9 \end{aligned}$$

which gives $\text{sol}^{\underline{n}}$ set $\boxed{\{(3, 6, 9)\}}$

PROBLEM 9

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 10 \\ 2x_1 + 2x_2 + 4x_4 &= 4 \end{aligned}$$

$$\boxed{\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 10 \\ 4 \end{bmatrix}}_{b}}$$

PROBLEM 10

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 2 & 2 & 0 & 4 & 4 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 0 & -2 & 2 & -16 \end{array}\right] \xrightarrow{r_1 + \frac{1}{2}r_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 & -16 \end{array}\right]$$

$$\xrightarrow{r_2/-2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & -1 & 8 \end{array}\right] \longrightarrow \begin{array}{l} x_1 + x_2 + 2x_4 = 2 \\ x_3 - x_4 = 8 \end{array}$$

Set $x_2 = t$ and $x_4 = s$ for $s, t \in \mathbb{R}$ and observe we have sol^{n}'s $x_1 = 2 - t - 2s$ and $x_3 = 8 - s$, $x_2 = t$, $x_4 = s$ or we could just as well say: $x_2 = \alpha$, $x_4 = \beta$

$$\boxed{\begin{array}{l} \text{Sol}^{n} \text{ Set} = \{ (2 - \alpha - 2\beta, \alpha, 8 + \beta, \beta) \mid \alpha, \beta \in \mathbb{R} \} \\ \hline \text{Sol}^{n} \text{ Set} = \{ (2 - x_2 - 2x_4, x_2, 8 + x_4, x_4) \mid x_2, x_4 \in \mathbb{R} \} \end{array}}$$

<u>Remark</u>: I find using x_2, x_4 as the parameters is easier to follow, but, these are the same <u>sets</u>. Notice, it is our custom to use non-pivot columns as free variables (parameters). Here columns 1 and 3 were pivot columns.

PROBLEM 11

$$\left.\begin{array}{l} x + y + z = 3 \\ 2x + y + z = 5 \\ 3x - y + z = 0 \\ 2x + 2y + 2z = 7 \end{array}\right\} \longrightarrow \boxed{\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}}_{b}}$$

PROBLEM 12

at this pt it is clear $\nexists$ sol^{n}'s $\therefore$ $\boxed{\text{sol}^{n} \text{ set} = \emptyset}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 5 \\ 3 & -1 & 1 & 0 \\ 2 & 2 & 2 & 7 \end{array}\right] \xrightarrow[\substack{r_3 - 3r_1 \\ r_4 - 2r_1}]{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -4 & -2 & -9 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - 4r_2]{r_1 + r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & 1 \end{array}\right] (\star) \xrightarrow{r_2 + \frac{1}{2}r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -7/2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{r_2 + \frac{7}{2}r_4 \\ r_3 + 5r_4}]{r_1 - 2r_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3/2]{-r_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\text{rref}[A|b].}$$

<u>Remark</u>: I did not <u>ask</u> for rref $[A|b]$. We could <u>logically</u> stop at $\star$

$\boxed{\text{PROBLEM 13}}$ List all 2×4 rref patterns:

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{PROBLEM 14}}$

$$\begin{bmatrix} x^2 & y \\ y^2 & x \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 9 & 2 \end{bmatrix} \begin{array}{l} \xrightarrow{11} x^2 = 4 \Rightarrow x = \pm 2 \\ \xrightarrow{12} y = -3 \\ \xrightarrow{21} y^2 = 9 \Rightarrow y = \pm 3 \\ \xrightarrow{22} x = 2 \end{array}$$

We need all 4 components to match $\therefore \boxed{x = 2,\ y = -3}$

($y = 3$ and $x = -2$ must be discarded due to 12 & 22 eq$^{\text{n}}$'s)

$\boxed{\text{PROBLEM 15}}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \underline{aE_{11} + bE_{12} + cE_{21} + dE_{22}}.$$

$\boxed{\text{PROBLEM 16}}$ Find $f(x) = Ax^2 + Bx + C$ which has $(1,1), (2,1)(3,1) \in \text{graph}(f)$

(algebra: $f(x) - 1 = 0$ for $x = 1, 2, 3 \Rightarrow f(x) = 1$ (degenerate poly. of 2$^{\text{nd}}$ order.))

$$\begin{aligned} f(1) &= A + B + C = 1 \\ f(2) &= 4A + 2B + C = 1 \\ f(3) &= 9A + 3B + C = 1 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 9 & 3 & 1 & 1 \end{array}\right] \xrightarrow[r_3 - 9r_1]{r_2 - 4r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & -3 \\ 0 & -6 & -8 & -8 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{6r_1} \\ \xrightarrow{-3r_2} \\ \xrightarrow{-r_3} \end{array} \left[\begin{array}{ccc|c} 6 & 6 & 6 & 6 \\ 0 & 6 & 9 & 9 \\ 0 & 6 & 8 & 8 \end{array}\right] \xrightarrow[r_3 - r_2]{r_1 - r_2} \left[\begin{array}{ccc|c} 6 & 0 & -3 & -3 \\ 0 & 6 & 9 & 9 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow[r_2 + 9r_3]{r_1 - 3r_3} \left[\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

oh, enough, we find $6A = 0$, $6B = 0$, $-C = -1$ $\therefore$ $A = 0, B = 0, C = 1$

$\therefore \boxed{f(x) = 1}$

PROBLEM 17 Find all polynomials whose graphs contain $(1,1)$, $(2,1)$, $(3,1)$.

If $f(1)=1$, $f(2)=1$, $f(3)=1$ then $g(x)=f(x)-1$ has $g(1)=0$, $g(2)=0$, $g(3)=0$ hence by factor Th$^{\underline{m}}$

$$g(x) = (x-1)(x-2)(x-3)h(x)$$ where $h(x)$ is a polynomial.

$\therefore$ $f(x) = A(x-1)(x-2)(x-3)+1$ at $\deg(f)=3$

$f(x) = (Ax+B)(x-1)(x-2)(x-3)+1$ for $\deg(f)=4$

Generally, $f(x) = h(x)(x-1)(x-2)(x-3)+1$ for $h(x)$ a polynomial.

<u>Remark</u>: I made announcement that 80% credit for sol$^{\underline{n}}$ of $\deg(f)=3$ would be given. I solved this problem with plain old algebra, perhaps you used linear algebra?

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<u>Linear algebra sol$^{\underline{n}}$</u>: consider $f(x) = a_nx^n + a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$

$$\left.\begin{aligned}
&(I)\ \ f(1) = a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 = 1\\
&(III)\ \ f(3) = 3^na_n + 3^{n-1}a_{n-1} + \cdots + 9a_2 + 3a_1 + a_0 = 1\\
&(II)\ \ f(2) = 2^na_n + 2^{n-1}a_{n-1} + \cdots + 4a_2 + 2a_1 + a_0 = 1
\end{aligned}\right\}$$

3 eq^{ns} in n-unknowns $\hookrightarrow$ possibly $(n-3)$ or more parameters in sol$^{\underline{n}}$. (at <u>least</u> $n-3$ parameters)

Note, (II) $-$ (I) : $(2^n-1)a_n + (2^{n-1}-1)a_{n-1}+\cdots+3a_2+a_1 = 0$

(III) $-$ (I) : $(3^n-1)a_n + (3^{n-1}-1)a_{n-1}+\cdots+8a_2+2a_1 = 0$

Next, subtract the equations above after multiplying (II) $-$ (I) by 2, That is calculate $2((II)-(I)) - [(III)-(I)]$,

$$[2(2^n-1)-(3^n-1)]a_n + [2(2^{n-1}-2)-(3^{n-1}-1)]a_{n-1}+\cdots+[6-8]a_2 = 0$$

Hence, $a_2 = \frac{1}{2}[2(2^n-1)-(3^n-1)]a_n + \frac{1}{2}[2(2^{n-1}-2)-(3^{n-1}-1)]a_{n-1}+\cdots$

this gives $a_2 = f_2(a_n, a_{n-1}, \dots, a_3)$ then (II) $-$ (I) allows us to write $a_1 = -3a_2 - (2^n-1)a_n - (2^{n-1}-1)a_{n-1}\cdots$ thus $a_1 = f_1(a_n, \dots, a_3)$.

PROBLEM 17

Clearly we have a_0, a_1, a_2 as functions of the remaining parameters $a_3, a_4, \ldots, a_n$. That said, in my current approach, it is not so clear how to see $f(x) = h(x)(x-1)(x-2)(x-3) + 1$ for some $h(x) = b_{n-3}x^{n-3} + \cdots + b_2 x^2 + b_1 x + b_0$. In fact the coefficients $b_0, b_1, \ldots, b_{n-3}$ relate to $a_3, a_4, \ldots, a_n$ in a rather complicated fashion. Long story short, highschool algebra won this time.

PROBLEM 18

$$\begin{aligned} \sin\alpha + \cos\beta + 2\sin\gamma &= \sqrt{2}+1 \\ 2\sin\alpha + 3\cos\beta - \sin\gamma &= 1-\sqrt{2} \\ 3\sin\alpha + \cos\beta &= 3 \end{aligned}$$

$$\left\{ \begin{aligned} x &= \sin\alpha \\ y &= \cos\beta \\ z &= \sin\gamma \end{aligned} \right\}$$

$$\begin{aligned} x+y+2z &= \sqrt{2}+1 \\ 2x+3y-z &= 1-\sqrt{2} \\ 3x+y &= 3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1+\sqrt{2} \\ 2 & 3 & -1 & 1-\sqrt{2} \\ 3 & 1 & 0 & 3 \end{array}\right] \xrightarrow[r_3-3r_1]{r_2-2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1+\sqrt{2} \\ 0 & 1 & -5 & 1-\sqrt{2}-2(1+\sqrt{2}) \\ 0 & -2 & -6 & 3-3(1+\sqrt{2}) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1+\sqrt{2} \\ 0 & 1 & -5 & -1-3\sqrt{2} \\ 0 & -2 & -6 & -3\sqrt{2} \end{array}\right] \xrightarrow{r_3+2r_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1+\sqrt{2} \\ 0 & 1 & -5 & -1-3\sqrt{2} \\ 0 & 0 & -16 & -3\sqrt{2}+2(-1-3\sqrt{2}) \end{array}\right]$$

$$-16z = -9\sqrt{2} - 2$$

$$z = \frac{9\sqrt{2}+2}{16} \rightarrow \alpha = \sin^{-1}\left(\frac{9\sqrt{2}+2}{16}\right)$$

$\alpha \approx 67°$ or 1.169 radians

(there are more ?)

once get to here we can use back-substitution if we like. Next,

$$y - 5z = -1 - 3\sqrt{2}$$

$$y = 5z - 1 - 3\sqrt{2} = 5\left(\frac{9\sqrt{2}+2}{16}\right) - 1 - 3\sqrt{2} = \frac{-6-3\sqrt{2}}{16} \approx -0.6402$$

$$y = \cos\beta \longrightarrow \beta \cong 129.8° \text{ or } 2.266 \text{ radians}$$

PROBLEM 18 continued

$$x + y + 2z = 1 + \sqrt{2}$$

$$x = 1 + \sqrt{2} - y - 2z = 1 + \sqrt{2} + \frac{6 + 3\sqrt{2}}{16} - \frac{2(9\sqrt{2} + 2)}{16}$$

$$x = \frac{18 + \sqrt{2}}{16} \approx 1.213$$

$$x = \sin\gamma \longrightarrow \gamma = \sin^{-1}\left(\frac{18+\sqrt{2}}{16}\right) \approx \text{NO SUCH VALUE !}$$

Well, we are free of detailing the ∞ # of additional sol^{n}'s for y and z (well really for α, β $\exists$ ∞ly many values which give $\frac{9\sqrt{2}+2}{16} = z$ and $y = \frac{-6-3\sqrt{2}}{16}$, $z \cong 1.169 + 2\pi k$ for $k \in \mathbb{Z}$ etc...) ANYWAY, it matters not as $\nexists$ a sol^{n} of $\sin\gamma = \frac{18+\sqrt{2}}{16}$ $\therefore$ $\boxed{\text{NO SOL}^{\underline{n}} \text{ EXISTS}}$

(this is the "solution" here)

<u>Remark</u>: if I ask you to "solve" a particular eq$^{\underline{n}}$ this is understood to mean find a sol$^{\underline{n}}$ if it exists or to explain why the sol^{n} set is empty.

$\boxed{\text{PROBLEM 19}}$ Consider $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ list all elements in $\mathbb{Z}_2^{2\times 2}$, <u>there are 16 possibilities</u>:

$$\begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix}$$

$$\begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}$$

$$\begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{1} & \text{[illegible]} \end{bmatrix}$$

<u>Remark</u>: if you used $\mathbb{Z}_2 = \{0, 1\}$ then I'm ok with that ☺ just understand in this context, $1 + 1 = 0$.

$\boxed{\text{PROBLEM 20}}$

Consider $\mathbb{Z}/6\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

$$\bar{3} \cdot \begin{bmatrix} \bar{0} & \bar{1} & \bar{2} \\ \bar{3} & \bar{4} & \bar{5} \end{bmatrix} = \begin{bmatrix} \bar{3}\cdot\bar{0} & \bar{3}\cdot\bar{1} & \bar{3}\cdot\bar{2} \\ \bar{3}\cdot\bar{3} & \bar{3}\cdot\bar{4} & \bar{3}\cdot\bar{5} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{0} & \bar{3} & \bar{6} \\ \bar{9} & \overline{12} & \overline{15} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{0} & \bar{3} & \bar{0} \\ \bar{3} & \bar{0} & \bar{3} \end{bmatrix}$$

the strange thing is that $\bar{3} \neq \bar{0}$ yet $\bar{3}\cdot\bar{2} = \bar{0}$ so, the nonzero scalar multiplication caused zeros to appear where the components were initially nonzero. This is the phenomenon of <u>zero-divisors</u>.

~~It is part~~ $\mathbb{Z}/6\mathbb{Z}$ is <u>not</u> a field as zero divisors cannot be units, so $\bar{3}$ and $\bar{2}$ are <u>nonzero</u> elements in $\mathbb{Z}/6\mathbb{Z}$ for which no multiplicative inverse exists. That said, you can work with matrices built over $\mathbb{Z}/6\mathbb{Z}$ (or other more abstract "rings") you just have to be careful to avoid use of <u>cancellation</u> properties which are based on <u>field</u> properties ($\bar{3}\cdot\bar{2} = \bar{3}\cdot\bar{4} \not\Rightarrow \bar{2} = \bar{4}$ for example)