Same instructions as Mission 1. Thanks!

roblem 21 Your signature below indicates you have:

(a.) I read Chapter 3 of Cook's lecture notes:________________.

Problem 22 Let $A = \begin{bmatrix} 1 & 2 \\ 7 & 0 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. Calculate AB, BA, AA^T and B^6, or say multiplication not defined for appropriate case(s).

Problem 23 Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$. Let $e_1, e_2, e_3 \in \mathbb{R}^3$ denote the usual standard basis. Also, let $\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \in \mathbb{R}^4$ denote the usual standard basis. Calculate the following:

(a.) $A\bar{e}_1$

(b.) $e_3^T A$

(c.) $A[\bar{e}_1|\bar{e}_2]$

(d.) $[e_1|e_2]^T A$

Problem 24 Consider the system of equations:

$$(\textbf{I.})\ \ y_1 + 2y_2 + 3y_3 = 4 \qquad \& \qquad 4y_1 + 5y_2 + 6y_3 = 13.$$

Where there exist x_1, x_2 for which:

$$(\textbf{II.})\ \ x_1 + x_2 = y_1 \qquad \& \qquad 2x_1 + x_2 = y_2 \qquad \& \qquad x_1 + 3x_2 = y_3.$$

Let $b_y = [y_1, y_2, y_3]^T$ and $b = [4, 13]^T$. Find a matrix M such that equation (**I.**) reads $Mb_y = b$. Then find another matrix N such that equation (**II.**) reads $N[x_1, x_2]^T = b_y$. Finally, solve both systems by making the natural substitution.
This problem illustrates one of the ways we first discovered matrix multiplication: it arose naturally whenever you are trying to solve two linear equations where the solution of one equation feeds into the other equation

Problem 25 Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$. Show that $(A - I)^3 = 8I$ and use that identity to derive an inverse for A in terms of a polynomial in A. In a problem such as this, you should understand the notation I to mean $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 26 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$. Find 2×2 elementary matrices E_1, E_2, E_3 such that $E_3E_2E_1A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$. Also, while we're at it, what is $[\mathrm{Col}_1(A)|\mathrm{Col}_2(A)]^{-1}$? Can you read off the inverse without further calculation?

Problem 27 It is also possible to perform elementary column operations on a matrix A. Furthermore, if you perform a column operation on the identity matrix then the resulting matrix when multiplied on the **right** of A will perform that same column operation. For example, to swap columns 2 and 3 of $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$ I first swap columns 2 and 3 of $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to obtain $C_{2\leftrightarrow 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and you can calculate

$$AC_{2\leftrightarrow 3} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

Find column operation matrices $C_1, C_2, \cdots, C_n$ such that $AC_1C_2\cdots C_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
I found either $n = 3$ or $n = 5$ depending on how you like to arrange the steps. This type of calculation has further significance if you study abstract algebra. In particular, see Dummit and Foote on the Smith Normal Form. For us, it's just a good exercise for matrix arithmetic.

Problem 28 Suppose that $Av_1 = 3e_1$ and $Av_2 = e_2 + e_3$ and $Av_3 = e_2 - e_3$. If $A \in \mathbb{R}^{3\times 3}$ then find the inverse of A in terms of the given vectors v_1, v_2, v_3.

Problem 29 **Prove that matrix multiplication is associative**; that is, show $A(BC) = (AB)C$ for all multipliable A, B, C. In particular, the term *multipliable* means you are to assume there exist $p, q, r, s \in \mathbb{N}$ such that $A \in \mathbb{R}^{p\times q}$, $B \in \mathbb{R}^{q\times r}$ and $C \in \mathbb{R}^{r\times s}$.

Problem 30 Suppose $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times q}$. **Prove:** $(AB)^T = B^TA^T$.

Problem 31 Consider the fact $\mathrm{rref}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Give three systems of equations and their solution, or lack of a solution, as derived from the given row-reduction. Two of these systems share the same 3×2 coefficient matrix and the third has a 3×3 coefficient matrix.

Problem 32 Let $A = \begin{bmatrix} k & 4k-3 \\ 1 & k \end{bmatrix}$. Determine what condition(s) you need on the constant k in order that A be invertible. Given those condition(s) calculate A^{-1}.

Problem 33 Given that $A^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}$ calculate $(A^TB)^{-1}$.

Problem 34 Find the inverse matrix of $M = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

Problem 35 Suppose A is invertible and N is a matrix such that $N = ANA$ and $N \neq 0$ yet $N^2 = 0$. Find $(A+N)^{-1}$. *Hint: guess and check.*

Problem 36 Define the **commutator of square matrices** A **and** B by $[A, B] = AB - BA$. Prove the Jacobi identity,

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

Problem 37 We say A **is similar to** B if there exists some invertible matrix P such that $B = P^{-1}AP$. In such a case we also denote similarity by $A \sim B$.
Prove: If $A_1 \sim A_2$ and $B_1 \sim B_2$ then $[A_1, B_1] \sim [A_2, B_2]$.

Problem 38 Let $A, B \in \mathbb{R}^{n \times n}$. **Prove that if** $v^T Aw = v^T Bw$ **for all** $v, w \in \mathbb{R}$ **then** $A = B$.
Note: it is important that we can see you understand what the phrase "for all" means as you write your solution. Calculations without words will not suffice here (or usually). In this course we need to explain the math as we do it.

Problem 39 Suppose $Q(x, y, z) = x^2 + 4xy + 2xz + y^2 - z^2$. Let $v = [x, y, z]^T$ and find a symmetric matrix A for which $Q(v) = v^T Av$.

Problem 40 Let $M = \begin{bmatrix} \bar{1} & \bar{2} \\ \bar{k} & \bar{1} \end{bmatrix}$ be a matrix built over $\mathbb{Z}/p\mathbb{Z}$ where p is an odd prime. Find the condition on the integer k which is necessary for M^{-1} to exist.

MISSION 2 SOLUTION : LINEAR ALGEBRA

PROBLEM 22 Let $A = \begin{bmatrix} 1 & 2 \\ 7 & 0 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Calculate AB, BA, AA^T, B^6 or indicate the product is not defined.

$$AB = \begin{bmatrix} 1 & 2 \\ 7 & 0 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 14 & 7 \\ 10 & 9 \end{bmatrix}.$$

BA ← not multipliable.
$(2\times 2)(3\times 2)$

$$AA^T = \begin{bmatrix} 1 & 2 \\ 7 & 0 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 7 & 49 & 35 \\ 9 & 35 & 29 \end{bmatrix}.$$

$(3\times 2)(2\times 3)$

$$B^6 = B^2B^2\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = B^2B^2\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 16 & 32 \\ 0 & 16 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 64 & 192 \\ 0 & 64 \end{bmatrix}}$$

PROBLEM 23 $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

(a.) $A\bar{e}_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}}.$

Remark: I hope you appreciate the point of these...

(b.) $e_3^T A = \underline{[9, 10, 11, 12]}.$

(c.) $A[\bar{e}_1|\bar{e}_2] = [A\bar{e}_1|A\bar{e}_2] = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 9 & 10 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 9 & 10 \end{bmatrix}}.$

(d.) $[e_1|e_2]^T A = \left[\dfrac{e_1^T}{e_2^T}\right]A = \left[\dfrac{e_1^TA}{e_2^TA}\right] = \underline{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}}.$

PROBLEM 24

$$(\mathrm{I.})\quad \left.\begin{aligned} y_1 + 2y_2 + 3y_3 &= 4 \\ 4y_1 + 5y_2 + 6y_3 &= 13 \end{aligned}\right\} \hookrightarrow \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{b_y} = \underbrace{\begin{bmatrix} 4 \\ 13 \end{bmatrix}}_{b}$$

where $\exists\, x_1, x_2$ such that

$$(\mathrm{II.})\quad \left.\begin{aligned} x_1 + x_2 &= y_1 \\ 2x_1 + x_2 &= y_2 \\ x_1 + 3x_2 &= y_3 \end{aligned}\right\} \hookrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}}_{N} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{b_y}$$

$$M b_y = MN \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$$

$$MN = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 20 & 27 \end{bmatrix} \hookrightarrow (MN)^{-1} = \frac{1}{-24} \begin{bmatrix} 27 & -12 \\ -20 & 8 \end{bmatrix}$$

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (MN)^{-1} \begin{bmatrix} 4 \\ 13 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} -27 & 12 \\ 20 & -8 \end{bmatrix} \begin{bmatrix} 4 \\ 13 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 48 \\ -24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \cancel{\begin{bmatrix} 4/3 - 5/9 \\ 8/3 - 5/9 \\ 4/3 - 15/9 \end{bmatrix} = \begin{bmatrix} 7/9 \\ 19/9 \\ -1/3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} \qquad \boxed{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}}$$

PROBLEM 25 Consider $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$. We show $(A - I)^3 = 8I$,

$$A - I = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix} \longrightarrow (A - I)^2 = \begin{bmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

$$\longrightarrow (A - I)^3 = (A - I)^2 (A - I) = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I.$$

On the other hand,

$$\begin{aligned} 8I = (A - I)^3 &= (A - I)(A - I)(A - I) \\ &= (A - I)(A^2 - A - A + I) \\ &= (A - I)(A^2 - 2A + I) \\ &= A^3 - 2A^2 + A - A^2 + 2A - I \end{aligned}$$

$$9I = A^3 - 3A^2 + 3A$$

$$\therefore I = A\left(\tfrac{1}{9}(A^2 - 3A + 3I)\right)$$

$$\therefore \boxed{A^{-1} = \tfrac{1}{9}(A^2 - 3A + 3I)}$$

Polynomial in A.

$\boxed{\text{PROBLEM 26}}$ Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$. Find 2×2 elementary matrices E_1, E_2, E_3 such that $E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}}_{A} \xrightarrow{r_2 - 3r_1} \underset{E_1 A}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \end{bmatrix}} \xrightarrow{r_1 + r_2} \underset{E_2(E_1 A)}{\begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -3 \end{bmatrix}} \xrightarrow{-r_2} \underset{E_3(E_2(E_1 A))}{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}}$$

$$\underline{E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}. \quad \underline{E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}. \quad \underline{E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}.$$

Thus, $\underbrace{E_3 E_2 E_1}_{E} A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

Remark: it is possible to obtain different E_1, E_2, E_3 but, we must agree on $\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}^{-1}$

$$\Rightarrow \left[E\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \middle| E\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right]$$

$$\hookrightarrow \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}^{-1} = E = E_3 E_2 E_1$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \underline{\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}}.$$

$$= \underline{\begin{bmatrix} -2 & +1 \\ 3 & -1 \end{bmatrix}}. \leftarrow \text{sorry, a mess.}$$

Check Result: $\begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$\boxed{\text{PROBLEM 27}}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix} \xrightarrow{c_1 \leftrightarrow c_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{c_2 - c_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{c_3 - c_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{c_2/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{c_3 - 3c_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$\underline{C_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}}.$$

$$AC_1 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \longmapsto AC_1C_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

etc... $\boxed{AC_1C_2C_3C_4C_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}}$

Remark: again, choice of C_1, C_2, C_3, C_4, C_5 not unique.

PROBLEM 28 Suppose $Av_1 = 3e_1$ and $Av_2 = e_2 + e_3$ and $Av_3 = e_2 - e_3$ where $A \in \mathbb{R}^{3\times 3}$. Find A^{-1} in terms of v_1, v_2, v_3.

Let us begin by noticing $A(v_1/3) = e_1$. We Next need to solve $Av = e_2$ and $Aw = e_3$

Let $v = xv_1 + yv_2 + zv_3$ and $w = \alpha v_1 + \beta v_2 + \gamma v_3$

$$\begin{aligned} Av &= A(xv_1 + yv_2 + zv_3) \\ &= xAv_1 + yAv_2 + zAv_3 \\ &= x(3e_1) + y(e_2+e_3) + z(e_2 - e_3) \\ &= (3x, y+z, y-z) = \underbrace{(0,1,0)}_{e_2} \end{aligned}$$

$x = 0$

$y = 1 - z$

$y = z$

$\Rightarrow 1 - z = z$

$\therefore \underline{z = 1/2 = y}$.

$$\begin{aligned} Aw &= A(\alpha v_1 + \beta v_2 + \gamma v_3) \\ &= \alpha Av_1 + \beta Av_2 + \gamma Av_3 \\ &= \alpha 3e_1 + \beta(e_2+e_3) + \gamma(e_2 - e_3) \\ &= (3\alpha, \beta+\gamma, \beta-\gamma) = \underbrace{(0,0,1)}_{e_3} \end{aligned}$$

$\alpha = 0$

$\beta = -\gamma$

$\beta = 1 + \gamma$

$\Rightarrow -\gamma = 1 + \gamma$

$\Rightarrow \underline{\gamma = -1/2 = -\beta}$

Hence $v = \frac{1}{2}(v_2 + v_3)$ and $w = \frac{1}{2}(v_2 - v_3)$

$$\therefore \boxed{A^{-1} = \left[\frac{1}{3}v_1 \,\middle|\, \frac{1}{2}(v_2+v_3) \,\middle|\, \frac{1}{2}(v_2 - v_3)\right]}$$

Of course, you could just add & subtract

$$\begin{aligned} Av_2 &= e_2 + e_3 \\ Av_3 &= e_2 - e_3 \end{aligned}$$

$+\ A(v_2+v_3) = 2e_2 \longrightarrow A\cdot\frac{1}{2}(v_2+v_3) = e_2.$

$-\ A(v_2 - v_3) = 2e_3 \longrightarrow A\cdot\frac{1}{2}(v_2 - v_3) = e_3.$

PROBLEM 29 Show $A(BC) = (AB)C$ for $A \in \mathbb{R}^{p \times q}$, $B \in \mathbb{R}^{q \times r}$, $C \in \mathbb{R}^{r \times s}$

Consider the following for $1 \le i \le p$ and $1 \le j \le s$,

$$\big(A(BC)\big)_{ij} = \sum_{k=1}^{q} A_{ik}(BC)_{kj} \quad : \text{def}^{\underline{n}} \text{ of matrix mult.}$$

$$= \sum_{k=1}^{q} A_{ik} \sum_{l=1}^{r} B_{kl} C_{lj} \quad : \text{def}^{\underline{n}} \text{ of matrix mult.}$$

$$= \sum_{k=1}^{q} \sum_{l=1}^{r} A_{ik}(B_{kl} C_{lj}) \quad : \text{pulled constant } A_{ik} \text{ into sum over } l \text{ (constant w.r.t. } l\text{)}$$

$$= \sum_{l=1}^{r} \left(\sum_{k=1}^{q} A_{ik} B_{kl} \right) C_{lj} \quad : \text{properties of finite sums and associativity of real \# multiplication.}$$

$$= \sum_{l=1}^{r} (AB)_{il} C_{lj} \quad : \text{def}^{\underline{n}} \text{ of matrix mult.}$$

$$= \big((AB)C\big)_{ij} \quad : \text{def}^{\underline{n}} \text{ of matrix mult.}$$

Thus, $A(BC) = (AB)C$ as the above holds $\forall i, j$. //

PROBLEM 30 Show $(AB)^T = B^T A^T$, for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times q}$

$$\big((AB)^T\big)_{ij} = (AB)_{ji} \quad : \text{def}^{\underline{n}} \text{ of transpose}$$

$$= \sum_{k=1}^{n} A_{jk} B_{ki} \quad : \text{def}^{\underline{n}} \text{ of matrix multiplication}$$

$$= \sum_{k=1}^{n} B_{ki} A_{jk} \quad : \text{real \# multiplication commutes.}$$

$$= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} \quad : \text{def}^{\underline{n}} \text{ of transpose}$$

$$= \big(B^T A^T\big)_{ij} \quad : \text{def}^{\underline{n}} \text{ of matrix mult.}$$

As the above holds for $1 \le i \le m$, $1 \le j \le q$ we conclude $(AB)^T = B^T A^T$. //

PROBLEM 31 It is given that $\text{rref}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ three systems to glean from this calculation are:

this happens to be false. Actually the last column should read $\begin{bmatrix} -6 \\ 11 \\ 4 \end{bmatrix}$

(a.) $\left.\begin{aligned} x + 2y + 3z &= 4 \\ 2x + 2y + 2z &= 2 \\ x + 2y + 4z &= 0 \end{aligned}\right\}$ has sol$^{\underline{n}}$ $\begin{aligned} x &= -2 \\ y &= 3 \\ z &= 0 \end{aligned}$

(b.) $\left.\begin{aligned} x + 2y &= 3 \\ 2x + 2y &= 2 \\ \underbrace{x + 2y}_{\text{ignores } 4^{th} \text{ column}} &= 4 \end{aligned}\right\}$ has no sol$^{\underline{n}}$ as third row is $0x + 0y = 1$ in reduced matrix.

(c.) $\left.\begin{aligned} x + 2y &= 4 \\ 2x + 2y &= 2 \\ \underbrace{x + 2y}_{\text{ignores } 3^{rd} \text{ column.}} &= 0 \end{aligned}\right\}$ has $\underline{\underline{\text{NO}}}$ sol$^{\underline{n}}$ ~~$x = -2$, $y = 3$~~

Remark: grader, $\exists$ other interpretations, you must think...

PROBLEM 32 $A = \begin{bmatrix} k & 4k - 3 \\ 1 & k \end{bmatrix}$. Determine what condition(s) are needed for k in order for A^{-1} to exist, in that context calculate A^{-1}.

There are several ways to capture the invertibility of A, we found $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ in Lecture etc... so, you can see we require $ad - bc \neq 0$. For the given A this indicates $k^2 - 4k + 3 \neq 0 \Rightarrow (k-1)(k-3) \neq 0$ thus we require $\underline{k \neq 1 \text{ and } k \neq 3}$. In case $k \neq 1, 3$ we find

$$\boxed{A^{-1} = \frac{1}{k^2 - 4k + 3}\begin{bmatrix} k & -(4k-3) \\ -1 & k \end{bmatrix}}$$

I investigate other argument for P32

[PROBLEM 32 continued]

Another criteria for A^{-1} existing is $A\vec{x}=0 \Leftrightarrow \vec{x}=0$.
Consider, for $k \neq 0$,

$$\left[\begin{array}{cc|c} k & 4k-3 & 0 \\ 1 & k & 0 \end{array}\right] \xrightarrow{kr_2} \left[\begin{array}{cc|c} k & 4k-3 & 0 \\ k & k^2 & 0 \end{array}\right] \xrightarrow{r_2-r_1} \left[\begin{array}{cc|c} k & 4k-3 & 0 \\ 0 & k^2-4k+3 & 0 \end{array}\right]$$

or, $(k^2-4k+3)y = 0 \Rightarrow y=0$ provided $k^2-4k+3 = (k-1)(k-3) \neq 0$

then $y=0 \Rightarrow 1^{st}$ eq^n $kx+(4k-3)y=0$ reduces to $kx=0$. But, we assumed $k\neq 0$ hence $x=0$.

Therefore, for $k \neq 0,1,3$ we have $A\vec{x}=0 \Leftrightarrow \vec{x}=0$.

Recall the $\Leftarrow$ direction always true for arbitrary A.

In the case $k=1$,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \; : \; \text{note } A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence the sol^n set of $A\vec{x}=0$ is not merely $\{0\}$.

In the case $k=3$,

$$A = \begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix} \; : \; \text{note } A\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -9+9 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $A\vec{x}=0 \not\Rightarrow \vec{x}=0$ (I used $\vec{x}$ to distinguish from scalar x in this sol^n)

Finally, the $k=0$ case,

$$A = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} \Rightarrow A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} y=0 \\ x=0 \end{array}$$

Thus $A\vec{x}=0 \Leftrightarrow \vec{x}=0$ and we find A^{-1} does exist in this case.

$$\left[\begin{array}{cc|cc} k & 4k-3 & 1 & 0 \\ 1 & k & 0 & 1 \end{array}\right] \xrightarrow{kr_2} \left[\begin{array}{cc|cc} k & 4k-3 & 1 & 0 \\ k & k^2 & 0 & k \end{array}\right] \xrightarrow{r_2-r_1} \left[\begin{array}{cc|cc} k & 4k-3 & 1 & 0 \\ 0 & k^2-4k+3 & -1 & k \end{array}\right]$$

$$\xrightarrow{\frac{r_2}{k^2-4k+3}} \left[\begin{array}{cc|cc} k & 4k-3 & 1 & 0 \\ 0 & 1 & \underbrace{\frac{-1}{k^2-4k+3}}_{\alpha} & \frac{k}{\alpha} \end{array}\right] \xrightarrow{r_1-(4k-3)r_2} \left[\begin{array}{cc|cc} k & 0 & 1+\frac{4k-3}{\alpha} & -\frac{(4k-3)k}{\alpha} \\ 0 & 1 & -1/\alpha & k/\alpha \end{array}\right]$$

$$\xrightarrow{r_1/k} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{k}\left[1+\frac{1}{\alpha}(4k-3)\right] & \frac{-4k+3}{\alpha} \\ 0 & 1 & -1/\alpha & k/\alpha \end{array}\right]$$

But, $1+\frac{4k-3}{k^2-4k+3} = \frac{k^2-\cancel{4k}+\cancel{3}+\cancel{4k}-\cancel{3}}{k^2-4k+3}$

$$\Rightarrow A^{-1} = \frac{1}{k^2-4k+3}\begin{bmatrix} k & -4k+3 \\ -1 & k \end{bmatrix}$$

PROBLEM 33 Given that $A^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}$, Calculate $(A^T B)^{-1}$

$$
\begin{aligned}
(A^T B)^{-1} &= B^{-1} (A^T)^{-1} \\
&= B^{-1} (A^{-1})^T \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}^T \\
&= \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} -2 & 2 \\ 18 & 22 \end{bmatrix}}.
\end{aligned}
$$

Note: $AA^{-1} = I$

$\Rightarrow (AA^{-1})^T = I^T = I$

$\Rightarrow (A^{-1})^T A^T = I$

$\therefore \underline{(A^T)^{-1} = (A^{-1})^T}$.

PROBLEM 34 Find the inverse matrix of $M = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$$
\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]
\xrightarrow{r_2 - 2r_1}
\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]
\xrightarrow{r_3 - 2r_2}
\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 9 & 4 & -2 & 1 \end{array}\right]
$$

$$
\begin{array}{c} \xrightarrow{18r_1} \\ \xrightarrow{9r_2} \\ \xrightarrow{4r_3} \end{array}
\left[\begin{array}{ccc|ccc} 18 & 0 & 36 & 18 & 0 & 0 \\ 0 & 9 & -36 & -18 & 9 & 0 \\ 0 & 0 & 36 & 16 & -8 & 4 \end{array}\right]
\begin{array}{c} \xrightarrow{r_1 - r_3} \\ \xrightarrow{r_2 + r_3} \\ \end{array}
\left[\begin{array}{ccc|ccc} 18 & 0 & 0 & 2 & 8 & -4 \\ 0 & 9 & 0 & -2 & 1 & 4 \\ 0 & 0 & 36 & 16 & -8 & 4 \end{array}\right]
$$

$$
\begin{array}{c} \xrightarrow{r_1/18} \\ \xrightarrow{r_2/9} \\ \xrightarrow{r_3/36} \end{array}
\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/18 & 8/18 & -4/18 \\ 0 & 1 & 0 & -2/9 & 1/9 & 4/9 \\ 0 & 0 & 1 & 16/36 & -8/36 & 4/36 \end{array}\right]
$$

$$
M^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 4 & -2 \\ -2 & 1 & 4 \\ 4 & -2 & 1 \end{bmatrix}
$$

I <u>usually</u> <u>check</u> my answer on these (gulp)

$$
MM^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & -2 \\ -2 & 1 & 4 \\ 4 & -2 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

(phew.)

PROBLEM 35 Suppose A is invertible and N is a matrix such that $N = ANA$ and $N \neq 0$ yet $N^2 = 0$. Find $(A+N)^{-1}$.

We seek B such that $B(A+N) = I$

Thus $BA = I - BN \Rightarrow B = (I - BN)A^{-1}$

$\Rightarrow B = A^{-1} - BAN$

$N = ANA$

$\hookrightarrow NA^{-1} = AN.$

$\Rightarrow B(I + AN) = A^{-1}$

$\Rightarrow B(A + ANA) = I$

$\Rightarrow B(A+N) = I$ (the circle is complete, but, useless.)

Consider, instead, a guess ☺

$$(A+N)(A^{-1} - N) = AA^{-1} - AN + NA^{-1} - N^2$$
$$= I - \cancel{AN} + \cancel{AN} - \underbrace{N^2}_{0}$$
$$= I.$$

oh, so we learned something in the circle

Thus, $\boxed{(A+N)^{-1} = A^{-1} - N}$

PROBLEM 36 Let $[A,B] = AB - BA$, prove Jacobi Identity,

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0.$$

Consider,

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] =$$
$$= [A, BC - CB] + [B, CA - AC] + [C, AB - BA]$$
$$= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C$$
$$= \underbrace{ABC}_{1} - \underbrace{ACB}_{2} - \underbrace{BCA}_{3} + \underbrace{CBA}_{4} + \underbrace{BCA}_{3} - \underbrace{BAC}_{5} - CAB + \underbrace{ACB}_{2} + CAB - \underbrace{CBA}_{4} - \underbrace{ABC}_{1} + \underbrace{BAC}_{5}$$
$$= 0.$$

$\boxed{\text{PROBLEM 37}}$ $A \sim B$ iff $\exists P$ such that $B = P^{-1}AP$.
Suppose $A_1 \sim A_2$ and $B_1 \sim B_2$, we seek to show $[A_1, B_1] \sim [A_2, B_2]$

By assumption, $\exists P, Q \in \mathbb{R}^{n\times n}$ such that $A_2 = P^{-1}A_1P$ and $B_2 = Q^{-1}B_1Q$. Consider,

$$\begin{aligned}
[A_2, B_2] &= A_2B_2 - B_2A_2 \\
&= P^{-1}A_1PQ^{-1}B_1Q - Q^{-1}B_1QP^{-1}A_1P \quad \text{need } P = Q \\
&= P^{-1}A_1B_1P - P^{-1}B_1A_1P \\
&= P^{-1}(A_1B_1 - B_1A_1)P \\
&= P^{-1}[A_1, B_1]P \quad \therefore \underline{[A_1, B_1] \sim [A_2, B_2]}.
\end{aligned}$$

<u>Remark</u>: it is important that $A_1 \sim A_2$ and $B_1 \sim B_2$ by the <u>same</u> similarity transformation $A \mapsto P^{-1}AP$.

$\boxed{\text{PROBLEM 38}}$ Let $A, B \in \mathbb{R}^{n\times n}$. Suppose $v^TAw = v^TBw \ \forall v, w \in \mathbb{R}^n$. Let $v = e_i$ and $w = e_j$ for some $i, j \in \mathbb{N}_n$. We proved in Lecture (notes etc.) that $e_i^TAe_j = A_{ij}$ and $e_i^TBe_j = B_{ij}$ $\therefore$ $A_{ij} = B_{ij}$ for all $i, j \in \mathbb{N}_n$ hence $A = B$.

<u>Don't believe it?</u> Well $A = \sum_{i,j=1}^{n} A_{ij}E_{ij}$ where $E_{ij} = e_ie_j^T$

thus
$$\begin{aligned}
e_k^TAe_l &= e_k^T\left(\sum_{i,j=1}^{n} A_{ij}e_ie_j^T\right)e_l \\
&= \sum_{i,j=1}^{n} A_{ij} \underbrace{e_k^Te_i}_{\delta_{ki}} \underbrace{e_j^Te_l}_{\delta_{jl}} = A_{kl}.
\end{aligned}$$

PROBLEM 39 Suppose $Q(x,y,z) = x^2+4xy+2xz+y^2-z^2$. Let $v=[x,y,z]^T$ and find symmetric matrix $A=A^T$ such that $Q(v) = v^TAv$

$$Q(x,y,z) = [x,y,z]\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x \; y \; z]\begin{bmatrix} ax+by+cz \\ bx+dy+ez \\ cx+ey+fz \end{bmatrix}$$

$$= ax^2+bxy+cxz+bxy+dy^2+eyz+cxz+eyz+fz^2$$

$$= ax^2+dy^2+fz^2+2bxy+2cxz+2eyz.$$

So, just compare to $Q(x,y,z)$ given to see $\boxed{A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}}$

We can check my answer,

$$[x,y,z]\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x,y,z]\begin{bmatrix} x+2y+z \\ 2x+y \\ x-z \end{bmatrix} = x(x+2y+z)+y(2x+y)+z(x-z)$$

$$= \underline{x^2+4xy+2xz+y^2-z^2}.$$

PROBLEM 40 (to ~~grad~~ grader, be generous here!) Let $M = \begin{bmatrix} \bar{1} & \bar{2} \\ \bar{k} & \bar{1} \end{bmatrix}$ be a matrix with entries in $\mathbb{Z}/p\mathbb{Z}$ where p is an odd prime. Find condition on $k \in \mathbb{Z}$ needed in order that M^{-1} exist.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad-bc)^{-1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

this formula still works in $\mathbb{Z}/p\mathbb{Z}$ provided we understand $(ad-bc)^{-1}$ to notate the multiplicative inverse modulo p. It follows a sufficient condition is given by $\bar{1}\bar{1} - \bar{2}\bar{k} \neq \bar{0}$. That is, we have M^{-1} exists if given $\overline{1-2k} \neq \bar{0}$. That is, $\nexists j \in \mathbb{Z}$ such that $1-2k = jp$. Equivalently, we need $jp+2k=1$ to have no sol^{n} in $\mathbb{Z}$. Note $\gcd(p,2)=1$ thus sol^{n}'s exist!

PROBLEM 40 continued

$\overline{1-2k} \neq \overline{0} \iff 1-2k \neq pj \quad \forall j \in \mathbb{Z}.$

Thus, if $\exists j \in \mathbb{Z}$ such that $1-2k = pj$ then $\overline{1-2k} = \overline{0}$

Notice, $pj + 2k = 1$ has sol^{n} if $\gcd(p, 2) = 1$, but this means it is always the case that $\exists j \in \mathbb{Z}$ such that $pj + 2k = 1$ (for some $k \in \mathbb{Z}$)

Oh, sorry, I should focus this more, say $p = 3$ for example,

$$\underset{①}{M} = \begin{bmatrix} \overline{1} & \overline{2} \\ \overline{0} & \overline{1} \end{bmatrix}, \quad \underset{②}{M} = \begin{bmatrix} \overline{1} & \overline{2} \\ \overline{1} & \overline{1} \end{bmatrix}, \quad \underset{③}{M} = \begin{bmatrix} \overline{1} & \overline{2} \\ \overline{2} & \overline{1} \end{bmatrix}$$

are the only choices.

①
$$\underbrace{\begin{bmatrix} \overline{1} & \overline{2} \\ \overline{0} & \overline{1} \end{bmatrix}}_{\text{invertible.}} \begin{bmatrix} \overline{1} & \overline{-2} \\ \overline{0} & \overline{1} \end{bmatrix} = \begin{bmatrix} \overline{1} & \overline{-2} + \overline{2} \\ \overline{0} & \overline{1} \end{bmatrix} = \begin{bmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{bmatrix}$$

②
$$\begin{bmatrix} \overline{1} & \overline{2} \\ \overline{1} & \overline{1} \end{bmatrix} \begin{bmatrix} \overline{1} & \overline{-2} \\ \overline{-1} & \overline{1} \end{bmatrix} = \begin{bmatrix} \overline{1} - \overline{2} & \overline{0} \\ \overline{0} & \overline{-2} + \overline{1} \end{bmatrix} = \begin{bmatrix} \overline{-1} & \overline{0} \\ \overline{0} & \overline{-1} \end{bmatrix}$$

$$\hookrightarrow \underline{\begin{bmatrix} \overline{1} & \overline{2} \\ \overline{1} & \overline{1} \end{bmatrix}^{-1} = \begin{bmatrix} \overline{-1} & \overline{2} \\ \overline{1} & \overline{-1} \end{bmatrix}}.$$

③
$$\underbrace{\begin{bmatrix} \overline{1} & \overline{2} \\ \overline{2} & \overline{1} \end{bmatrix}}_{\text{not invertible}} \begin{bmatrix} \overline{1} & \overline{-2} \\ \overline{-2} & \overline{1} \end{bmatrix} = \begin{bmatrix} \overline{-3} & \overline{0} \\ \overline{0} & \overline{-3} \end{bmatrix} = \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}$$

for $p = 3$, $k \neq 2$

$$\overline{1-2k} = \overline{1-4} = \overline{-3} = \overline{0}.$$

Remark: grader, please let me know if someone improved on my sol^{n} here!