Same instructions as Mission 1. Thanks!

Problem 41 Your signature below indicates you have:

(a.) I read Chapters 4 and 5 of Cook's lecture notes: ________________.

Problem 42 Give a proof by mathematical induction that: for each $n \in \mathbb{N}$ the product of n-multipliable matrices satisfied the n-folded socks-shoes rule: $(A_1A_2\cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$.

Problem 43 Consider the equation $Av = b$ where:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} \qquad \& \qquad b = \begin{bmatrix} 3 \\ 9 \\ 11 \\ 12 \end{bmatrix}$$

You can easily see the equation $Av = b$ has no solutions. That said, solve $A^TAv = A^Tb$. *It turns out (for reasons we explain much later in this course) multiplication by A^T has the effect of removing the part of b which causes the original system to be inconsistent. The meaning of this problem is to find the least-squares fit of m, c such that $y = mx + c$ is the closest to the data points* $(0,3), (1,9), (2,11)$ *and* $(2,12)$.

Problem 44 Let $Z = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ and find all real solutions of $Z^2 = I$.

Problem 45 Consider the block-matrices $M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $D = \left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right]$. Assume the blocks are square and have matching matrix dimensions. Given that $MD = DM$ which blocks of M commute with D_1 ? Also, which blocks of M commute with D_2 ?

Problem 46 Show that the product of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 0 & 5 & 1 \end{bmatrix}$ and $v = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is a linear combination of vectors; $Av = c_1w_1 + c_2w_2 + c_3w_3$. Identify the vectors w_1, w_2, w_3 as they relate to A.

Problem 47 Is $(1,2,0,4) \in \text{span}\{e_1 + 2e_3, 2e_2 + e_3 + e_4, 3e_1 - e_2 - e_3 - e_4\}$?

Problem 48 Is $v = (a,b,c,d) \in \text{span}\{(1,2,3,0),(0,4,4,1)\}$? What condition(s) must be made on a, b, c, d for v to be in the span? Write down the set of vectors which is not in the span.

Problem 49 Are $(1,2,3)$ and $(0,1,1)$ in $\text{span}\{(1,3,4),(-1,3,4)\}$? If so, explicitly give the linear combinations to prove both of your assertions.

Problem 50 For what value(s) of k is the set S of vectors below a linearly independent set ?

$$S = \{(1,2,3),(2,2,2),(3,4,k)\}.$$

Problem 51 Let $S = \{v_1, v_2\} \subset \mathbb{R}^n$ and $T = \{v_1, v_2, v_3\} \subset \mathbb{R}^n$. Suppose that S is a linearly independent set. Is $S \cup T$ linearly indendent ? What about $S \cap T$? Prove your assertions.

Problem 52 Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 1 & -1 \\ 2 & 1 & 3 & 0 & 0 \\ 2 & 3 & 5 & 1 & 4 \end{bmatrix}$ and after some calculation we find:

$$\text{rref}\begin{bmatrix} 1 & 2 & 3 & 1 & -1 \\ 2 & 1 & 3 & 0 & 0 \\ 2 & 3 & 5 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & -5 \\ 0 & 1 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & -16 \end{bmatrix}.$$

If we denote $A = [v_1|v_2|v_3|v_4|v_5]$ then determine all subsets of $\{v_1, v_2, v_3, v_4, v_5\}$ which have three elements and are linearly independent. *(please use the CCP to guide your selection process: here CCP standard for the Column Correspondence Property which is sometimes known as the Linear Correspondence)*

Problem 53 You have two sets of vectors $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2\}$ in $\mathbb{R}^5$. Let $[T|S]$ denote the matrix formed by listing the vectors in S and T as columns. Furthermore,

$$\text{rref}[S|T] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Use the CCP to decide if S and T form LI sets. Also, determine which vectors in T fall inside span(S).

Problem 54 Find the standard matrix of $T : \mathbb{R}^3 \to \mathbb{R}^2$ which is defined by

$$T(x, y, z) = (2x + 3y - z, x + y + z).$$

Problem 55 Find a linear transformation T for which $T(1, 2, 2) = (0, 1, 0)$ and $T(3, 0, 3) = (1, 1, 0)$ and $T(0, 0, 1) = (1, 2, 3)$. You should give both the formula for T as well as the standard matrix.

Problem 56 Let $\mathcal{P} = \text{span}\{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$. We define an **affine space** to be a space $\mathcal{A}$ of the form $\mathcal{A} = x_o + \mathcal{P} = \{x_o + v \mid v \in \mathcal{P}\}$ where $x_o \in \mathbb{R}^n$. The point x_o is called a **base-point** of $\mathcal{A}$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that the image under T of an affine space is once again an affine space; that is, given $\mathcal{A}$ is an affine space show that $T(\mathcal{A})$ is an affine space.

Problem 57 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(e_1) = (1, 1, 2)$ and $T(e_2) = (2, 2, 1)$ and $T(e_3) = (1, 0, 1)$. Show that T is both injective and surjective. Find the formula for T^{-1}.

Problem 58 Suppose T is a rotation about the z-axis by θ and S is a rotation about the y-axis by β. Find the standard matrix for $T \circ S$ and $S \circ T$.

Problem 59 Let $T(v) = Av$ be a mapping from $\mathbb{R}^n \to \mathbb{R}^n$ for which $A^T = A$. Find how $v \bullet T(w)$ and $T(v) \bullet w$ relate for $v, w \in \mathbb{R}^n$. If necessary, break into cases.

Problem 60 And now for something a bit different: if $A : \mathbb{R} \to \mathbb{R}^{m\times n}$ defines a matrix-valued function of a real-variable then $\left(\frac{dA}{dt}\right)_{ij} = \frac{dA_{ij}}{dt}$ defines the derivative of such a function (provided the derivatives of each component function A_{ij} exist of course). Show the product rule works in this context: that is, given two differentiable multipliable matrix-valued functions A, B we have

$$\frac{d}{dt}(AB) = \frac{dA}{dt}B + A\frac{dB}{dt}.$$

Notice: the products above are matrix-products. Your argument ought to use the index-arithmetic and some plain-old single-variable calculus. *the calculus of matrix-valued functions is a fun topic which we continue to revisit from time to time.*

MISSION 3 SOLUTION : LINEAR ALGEBRA

P42 Suppose $A_1, A_2, \dots$ are multipliable matrices. Observe

$(A_1 A_2)^T = A_2^T A_1^T$ by socks-shoes prop. (which we proved previously)

Assume $(A_1 A_2 \cdots A_n)^T = A_n^T A_{n-1}^T \cdots A_2^T A_1^T$ for some $n \in \mathbb{N}$.

Consider multipliable matrices $A_1, A_2, \dots, A_{n+1}$,

$$\begin{aligned}(A_1 A_2 \cdots A_n A_{n+1})^T &= \left[(A_1 A_2 \cdots A_n)(A_{n+1})\right]^T && : \text{associativity of matrix multiplication.} \\ &= A_{n+1}^T (A_1 A_2 \cdots A_n)^T && : \text{socks-shoes for 2.} \\ &= A_{n+1}^T A_n^T \cdots A_2^T A_1^T && : \text{by induction hypothesis.}\end{aligned}$$

Hence n-socks-shoes $\Rightarrow$ $(n+1)$-socks-shoes $\therefore$ socks-shoes identity holds $\forall n \in \mathbb{N}$.

P43 Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 9 \\ 11 \\ 12 \end{bmatrix}$. Solve $A^TAv = A^Tb$.

$$A^TA = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 5 & 4 \end{bmatrix} \hookrightarrow (A^TA)^{-1} = \frac{1}{36-25}\begin{bmatrix} 4 & -5 \\ -5 & 9 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{11}\begin{bmatrix} 4 & -5 \\ -5 & 9 \end{bmatrix}$$

Observe $(A^TA)v = A^Tb \Rightarrow v = (A^TA)^{-1}A^Tb$

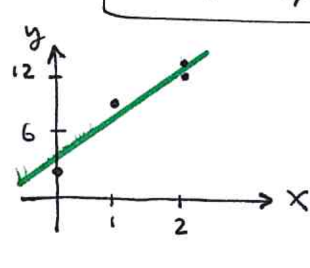
But, $A^Tb = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 55 \\ 35 \end{bmatrix}$

$$\hookrightarrow \quad v = \frac{1}{11}\begin{bmatrix} 4 & -5 \\ -5 & 9 \end{bmatrix}\begin{bmatrix} 55 \\ 35 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 45 \\ 40 \end{bmatrix} = \begin{bmatrix} 45/11 \\ 40/11 \end{bmatrix}$$

$\therefore$ $\boxed{v = (45/11, 40/11)}$

If you accept the claims of the problem statement then we've shown that $y = \frac{45}{11}x + \frac{40}{11}$ is closest line to the points $(0,3), (1,9), (2,11), (2,12)$

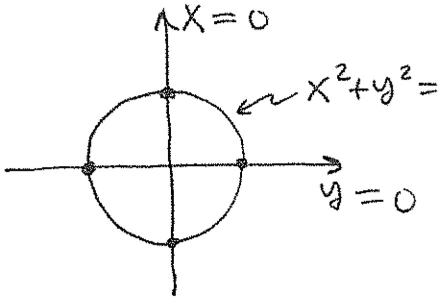
seems about right $\rightarrow$



P44 Let $Z = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ and find all <u>real</u> sol^{n}'s of $Z^2 = I$.

$$Z^2 = \begin{bmatrix} x & y \\ y & x \end{bmatrix}\begin{bmatrix} x & y \\ y & x \end{bmatrix} = \left[\begin{array}{c|c} x^2+y^2 & 2xy \\ \hline 2xy & y^2+x^2 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right]$$

We face the problem of solving $x^2+y^2=1$ and $2xy=0$. Geometrically, this amounts to intersection of the unit-circle and the coord. axes,



$2xy = 0 \Rightarrow x=0$ or $y=0$

If $x=0$ and $x^2+y^2=1 \Rightarrow y=\pm 1$

If $y=0$ and $x^2+y^2=1 \Rightarrow x=\pm 1$.

Thus, $x=0,\ y=\pm 1$ or $y=0,\ x=\pm 1$ are the allowed sol^{n}'s. This gives $\boxed{Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$.

P45 Consider $M = \left[\begin{array}{c|c} A & B \\ \hline C & \bar{D} \end{array}\right]$ and $D = \left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right]$ given all blocks are square and M & D have compatible blocks with $MD = DM$. Which blocks of M commute with D_1? What about D_2?

$$MD = \begin{bmatrix} A & B \\ C & \bar{D} \end{bmatrix}\left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right] = \left[\begin{array}{c|c} AD_1 & BD_2 \\ \hline CD_1 & \bar{D}D_2 \end{array}\right]$$

$$DM = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}\begin{bmatrix} A & B \\ C & \bar{D} \end{bmatrix} = \left[\begin{array}{c|c} D_1 A & D_1 B \\ \hline D_2 C & D_2 \bar{D} \end{array}\right]$$

$AD_1 = D_1 A$

$BD_2 = D_1 B$

$CD_1 = D_2 C$

$\bar{D}D_2 = D_2\bar{D}$

By equating blocks in $MD = DM$ we found $AD_1 = D_1 A$ and $\bar{D}D_2 = D_2\bar{D}$

Thus $\boxed{A \text{ commutes with } D_1 \text{ and } \bar{D} \text{ commutes with } D_2}$

$\boxed{P46}$ Show product of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 0 & 5 & 1 \end{bmatrix}$ and $v = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is linear combination of some vectors w_1, w_2, w_3 as $Av = c_1 w_1 + c_2 w_2 + c_3 w_3$. Identify w_1, w_2, w_3 as they relate to A.

$$Av = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 + 3c_3 \\ 4c_1 + 4c_2 + 4c_3 \\ 5c_2 + c_3 \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}}_{w_1} + c_2 \underbrace{\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}}_{w_2} + c_3 \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}}_{w_3}$$

Hence, a natural choice would be to see $w_1 = col_1(A)$, $w_2 = col_2(A)$ and $w_3 = col_3(A)$.

$\boxed{P47}$ Is $(1,2,0,4) \in \text{span}\{\underbrace{e_1 + 2e_3,\ 2e_2 + e_3 + e_4,\ 3e_1 - e_2 - e_3 - e_4}_{S}\}$?

A.k.a. can we solve $c_1(1,0,2,0) + c_2(0,2,1,1) + c_3(3,-1,-1,-1) = (1,2,0,4)$?

Hence, consider,

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 2 \\ 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 4 \end{array}\right] \xrightarrow{r_3 - 2r_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & -7 & -2 \\ 0 & 1 & -1 & 4 \end{array}\right] \xrightarrow{r_3 - r_4} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & -6 & -6 \\ 0 & 1 & -1 & 4 \end{array}\right]$$

$$\xrightarrow[r_3/-6]{r_2 - 2r_4} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 4 \end{array}\right] \xrightarrow{r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 7 \\ 0 & 1 & -1 & 4 \end{array}\right] \longrightarrow \text{oops } 0 = 7$$

$\therefore$ no sol$^{\underline{n}}$

$\Rightarrow \boxed{(1,2,0,4) \notin \text{span } S.}$

You can calculate that

$$\text{rref}\left[[S] \middle| \begin{matrix} 1 \\ 2 \\ 0 \\ 4 \end{matrix}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

But, there is no need. As soon as we reach an equivalent system to the initial system which is <u>manifestly</u> inconsistent we may stop the row-reduction and make our conclusion.

P48 Is $v = (a,b,c,d) \in \text{span}\{(1,2,3,0),(0,4,4,1)\}$?
What conditions are needed on a,b,c,d for v to be in the span?
Write down the <u>set</u> of vectors <u>not</u> in span(S), $S = \{(1,2,3,0),(0,4,4,1)\}$.

We seek $c_1, c_2 \in \mathbb{R}$ for which $c_1(1,2,3,0) + c_2(0,4,4,1) = (a,b,c,d)$
Hence, consider,

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 2 & 4 & b \\ 3 & 4 & c \\ 0 & 1 & d \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 4 & b-2a \\ 0 & 4 & c-3a \\ 0 & 1 & d \end{array}\right] \begin{array}{l} \longrightarrow c_1 = a \\ \longrightarrow 4c_2 = b-2a \\ \longrightarrow 4c_2 = c-3a \\ \longrightarrow c_2 = d \end{array}$$

$$\xrightarrow[r_3 - 4r_4]{r_2 - 4r_4} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 0 & b-2a-4d \\ 0 & 0 & c-3a-4d \\ 0 & 1 & d \end{array}\right]$$

} clearly consistent only when we have both
$b - 2a - 4d = 0$
and
$c - 3a - 4d = 0$

Thus, the set of vectors which are <u>not</u> in span(S) are

$$\boxed{\{(a,b,c,d) \mid a,d \in \mathbb{R} \text{ with } b \neq 2a+4d \text{ or } c \neq 3a+4d\}}$$

P49 Are $(1,2,3)$ $(0,1,1) \in \text{span}\{(1,3,4),(-1,3,4)\}$? If so, explicitly show the linear combinations which make it so.

Consider,

$$\left[\begin{array}{cc|c|c} 1 & -1 & 1 & 0 \\ 3 & 3 & 2 & 1 \\ 4 & 4 & 3 & 1 \end{array}\right] \xrightarrow[r_3-4r_1]{r_2-3r_1} \left[\begin{array}{cc|c|c} 1 & -1 & 1 & 0 \\ 0 & 6 & -1 & 1 \\ 0 & 8 & -1 & 1 \end{array}\right] \xrightarrow[\substack{4r_2 \\ 3r_3}]{24r_1} \left[\begin{array}{cc|cc} 24 & -24 & 24 & 0 \\ 0 & 24 & -4 & 4 \\ 0 & 24 & -3 & 3 \end{array}\right] \xrightarrow[r_2-r_3]{r_1+r_3} \left[\begin{array}{cc|cc} 24 & 0 & 21 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 24 & -3 & 3 \end{array}\right]$$

$$\xleftrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{cc|cc} 24 & 0 & 21 & 3 \\ 0 & 24 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

$\swarrow$ $\boxed{\text{neither } (1,2,3) \text{ or } (0,1,1) \text{ are in span}\{(1,3,4),(-1,3,4)\}.}$ (nor)

In contrast, $(0,6,8)$, $(-1,27,36) \in \text{span}\{(1,3,4),(-1,3,4)\}$ as is shown by

$$\text{rref}\left[\begin{array}{cc|c|c} 1 & -1 & 0 & -1 \\ 3 & 3 & 6 & 27 \\ 4 & 4 & 8 & 36 \end{array}\right] = \left[\begin{array}{cc|c|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow (1,3,4) + (-1,3,4) = (0,6,8).$$

$$4(1,3,4) + 5(-1,3,4) = (-1,27,36).$$

$\boxed{\text{P50}}$ For what value(s) of k is the set S of vectors $S = \{(1,2,3), (2,2,2), (3,4,k)\}$ a LI set?

We need $c_1(1,2,3) + c_2(2,2,2) + c_3(3,4,k) = (0,0,0) \Rightarrow c_1 = 0, c_2 = 0, c_3 = 0.$

Hence, consider,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 2 & k \end{bmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & -4 & k-9 \end{bmatrix} \xrightarrow[-r_3]{\substack{2r_1 \\ 2r_2}} \begin{bmatrix} 2 & 4 & 6 \\ 0 & -4 & -4 \\ 0 & 4 & 9-k \end{bmatrix}$$

$$\xrightarrow[r_2 + r_3]{r_1 - r_3} \begin{bmatrix} 2 & 0 & 6-(9-k) \\ 0 & 0 & -4+(9-k) \\ 0 & 4 & 9-k \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 2 & 0 & k-3 \\ 0 & 4 & 9-k \\ 0 & 0 & 5-k \end{bmatrix} \quad (*)$$

If $k \neq 5$ then the row-reduction continues to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and so we find S is LI as $\text{rref}[[S]|0] = \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \rightarrow \begin{matrix} c_1 = 0. \\ c_2 = 0. \\ c_3 = 0. \end{matrix}$

If $k = 5$ then $(*)$ yields,

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_2/4]{r_1/2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

3rd Column is sum of first and second column of $[S]$ by CCP. That is, $(1,2,3) + (2,2,2) = (3,4,5)$

In conclusion, $\boxed{S \text{ is LI if } k \neq 5}$

NOT LI for case $k = 5$.

$\boxed{\text{P51}}$ Let $S = \{V_1, V_2\} \subset \mathbb{R}^n$ and $T = \{V_1, V_2, V_3\} \subset \mathbb{R}^n$. Assume S is LI. Is $S \cup T$ a LI set? What about $S \cap T$? Prove your assertions

Observe, $S \cup T = \{V_1, V_2, V_3\}$ (viewing S and T as sets)

If $V_3 = V_1 + V_2$ then $S \cup T$ is not LI. However, if $V_3 \notin \text{span}\{V_1, V_2\}$ then we can argue $S \cup T$ is LI as follows,

$$c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$$

If $c_3 \neq 0$ then $V_3 = \frac{-1}{c_3}(c_1 V_1 + c_2 V_2) = -\frac{c_1}{c_3} V_1 - \frac{c_2}{c_3} V_2 \in \text{span}\{V_1, V_2\}$ hence $\rightarrow\leftarrow$ of $V_3 \notin \text{span}\{V_1, V_2\}$. Thus, $c_3 = 0$ and we find $c_1 V_1 + c_2 V_2 = 0 \Rightarrow c_1 = c_2 = 0$ by LI of $\{V_1, V_2\}$.

Thus $\{V_1, V_2, V_3\}$ is LI. (again, this assumes $V_3 \notin \text{span}\{V_1, V_2\}$)

So, in summary, it is POSSIBLE that $S \cup T$ is LI set.

continued discuss: $(S \cap T)$

P51 continued

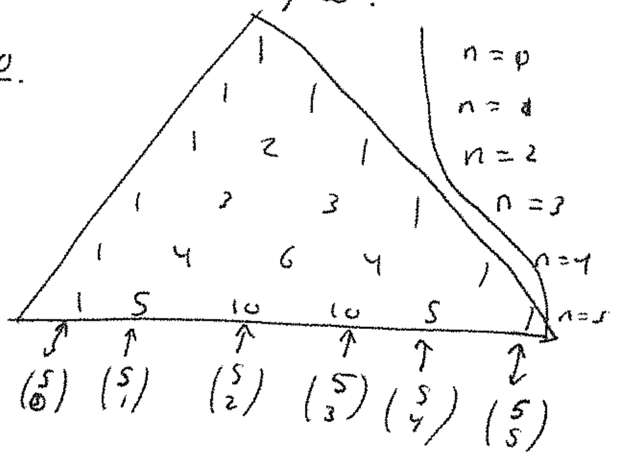
$$S \cap T = \{V_1, V_2\} \cap \{V_1, V_2, V_3\} = \{V_1, V_2\} = S$$

and we were given S is LI hence $S \cap T$ is LI.

P52 Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 1 & -1 \\ 2 & 1 & 3 & 0 & 0 \\ 2 & 3 & 5 & 1 & 4 \end{bmatrix}$ and you are given (correctly!) that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -5 \\ 0 & 1 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & -16 \end{bmatrix}$. Find all LI triples in the columns of A

How many ways to choose 3 from 5 in principle?

$$\binom{5}{3} = \frac{5!}{(5-3)!\,3!} = \frac{5\cdot 4\cdot 3\cdot 2}{(2\cdot 1)(3\cdot 2)} = 5\cdot 2 = \underline{10}.$$



We have ten possibilities to consider. For $A = [V_1|V_2|V_3|V_4|V_5]$ consider the following in view of CCP and given rref A,

① $\{V_1, V_2, V_3\}$ not LI as $V_3 = V_1 + V_2$.

② $\{V_1, V_2, V_4\}$ is LI by CCP.*

③ $\{V_1, V_2, V_5\}$ is LI by CCP.*

④ $\{V_2, V_3, V_4\}$ is LI by CCP*

⑤ $\{V_2, V_3, V_5\}$ is LI by CCP*

⑥ $\{V_2, V_4, V_5\}$ is LI by CCP*

⑦ $\{V_3, V_4, V_5\}$ is LI by CCP*

⑧ $\{V_1, V_3, V_4\}$ is LI by CCP*

⑨ $\{V_1, V_3, V_5\}$ is LI by CCP*

⑩ $\{V_1, V_4, V_5\}$ is LI by CCP*

*: it is clear from rref(A) that $\nexists$ a way to write one of the columns in rref(A) (the columns in question for the case considered) as a lin. combo. of the remaining two.

$\boxed{\text{Conclusion: all triples of columns from A are LI } \underline{\text{except}} \ \{V_1, V_2, V_3\} \text{ for which } V_3 = V_1 + V_2.}$

P53 Let $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2\}$ and form 7

$[T|S]$ by concatenating elements of S & T as columns of $[T|S]$

You're given that:

$$\text{rref}[S|T] = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Determine if S and T form LI sets (use CCP to guide)

Also, determine which vectors in T fall within span(S)

Yes, it is clear that $\boxed{S \text{ is LI}}$ as the 1^{st} three columns $\{e_1, e_2, e_3\}$ are clearly LI. Likewise, columns 4 and 5 are also forming a LI set as $(0,0,0,1,0) \neq k(4,3,2,0,0)$ for any $k \in \mathbb{R}$. Hence by CCP we find $\boxed{T \text{ is LI}}$.

Considering columns 1, 2, 3 and 4 we see inconsistent case thus $t_1 \notin \text{span}(S)$. On the other hand, considering columns 1, 2, 3 and 5 we see $\boxed{t_2 = 4s_1 + 3s_2 + 2s_3 \in \text{span}(S)}$

P54 $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(x,y,z) = (2x+3y-z, x+y+z)$. Find $[T]$.

Approach 1: $T(x,y,z) = \begin{bmatrix} 2x+3y-z \\ x+y+z \end{bmatrix} \overset{*}{=} \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ Since $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ knew to expect a (2×3) matrix $[T]$.

Hence $\boxed{[T] = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}}$. Here $*$ was by inspection or reverse-engineered matrix multiplication

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Approach 2: Use our Th$^{\underline{m}}$, T linear $\Rightarrow [T] = [T(e_1)|T(e_2)|T(e_3)]$

$$T(e_1) = T(1,0,0) = (2,1)$$
$$T(e_2) = T(0,1,0) = (3,1)$$
$$T(e_3) = T(0,0,1) = (-1,1)$$

$\therefore \boxed{[T] = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}}$

It's ALL GOOD.

$\boxed{\text{P55}}$ Find linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ for which $T(1,2,2) = (0,1,0)$, $T(3,0,3) = (1,1,0)$ and $T(0,0,1) = (1,2,3)$. Give a formula for T and $[T]$. (8)

My preference, find $[T]$ then build the formula. To find $[T] = [v_1|v_2|v_3]$ we need $T(e_1) = v_1$, $T(e_2) = v_2$, $T(e_3) = v_3$.

We're only given $T(e_3) = (1,2,3)$ which shows $[T] = \left[v_1 \,|\, v_2 \,|\, \begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]$

So, how to isolate e_1 and e_2 from our given inputs?

(1) Want to solve,

$$a(1,2,2) + b(3,0,3) + c(0,0,1) = (1,0,0)$$

$$a + 3b = 1$$
$$2a + 0 + 0 = 0 \Rightarrow a = 0 \Rightarrow 3b = 1 \therefore \underline{b = 1/3}$$
$$2a + 3b + c = 0 \Rightarrow 3(1/3) + c = 0 \therefore \underline{c = -1}.$$

That is, $\frac{1}{3}(3,0,3) - (0,0,1) = (1,0,0)$. Who cares? Why bother? Well, see below:

$$T(e_1) = T\left(\tfrac{1}{3}(3,0,3) - (0,0,1)\right) = \tfrac{1}{3}T(3,0,3) - T(0,0,1) = \tfrac{1}{3}\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-2/3\\-5/3\\-3\end{bmatrix}$$

Thus $\mathrm{col}_1([T]) = [-2/3, -5/3, -3]^T$.

(2) Solve $\bar{a}(1,2,2) + \bar{b}(3,0,3) + \bar{c}(0,0,1) = (0,1,0)$

Thus, $\bar{a} + 3\bar{b} = 0$, $2\bar{a} = 1$, $2\bar{a} + 3\bar{b} + \bar{c} = 0$. Thus, $\underline{\bar{a} = 1/2}$.
and $\bar{b} = -\bar{a}/3 = \underline{-1/6}$, and $\bar{c} = -2\bar{a} - 3\bar{b} = -1 + 3/6 = \underline{-1/2 = \bar{c}}$.

Thus,
$$\begin{aligned}
T(e_2) &= T\left(\tfrac{1}{2}(1,2,2) - \tfrac{1}{6}(3,0,3) - \tfrac{1}{2}(0,0,1)\right) \\
&= \tfrac{1}{2}T(1,2,2) - \tfrac{1}{6}T(3,0,3) - \tfrac{1}{2}T(0,0,1) \\
&= \tfrac{1}{2}(0,1,0) - \tfrac{1}{6}(1,1,0) - \tfrac{1}{2}(1,2,3) \\
&= (-1/6 - 1/2,\ 1/2 - 1/6 - 1,\ -3/2) = (-2/3, -2/3, -3/2)
\end{aligned}$$

Thus, $\boxed{[T] = \begin{bmatrix}-2/3 & -2/3 & 1\\ -5/3 & -2/3 & 2\\ -3 & -3/2 & 3\end{bmatrix}}$ continued → formula easy to derive from $[T]$, PSS continued

$$[T] = \begin{bmatrix} -2/3 & -2/3 & 1 \\ -5/3 & -2/3 & 2 \\ -3 & -3/2 & 3 \end{bmatrix}$$

$$\boxed{T(x,y,z) = [T]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(-\frac{2x}{3} - \frac{2y}{3} + z,\ \frac{-5x}{3} - \frac{2y}{3} + 2z,\ -3x - \frac{3y}{2} + 3z\right)}$$

Check work: $T(1,2,2) \overset{\checkmark}{=} \left(-\frac{2}{3} - \frac{4}{3} + 2,\ \frac{-5}{3} - \frac{4}{3} + 4,\ -3-3+6\right) = (0,1,0)$

$T(3,0,3) \overset{\checkmark}{=} (-2+3,\ -5+6,\ -9+9) = (1,1,0)$

$T(0,0,1) \overset{\checkmark}{=} (1,2,3)$.

Remark: there are other ways to attack (PSS). For example,

I.) Just set $[T] = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and solve 9 eq^n's an 9 unknowns that come from given data. Not elegant, ~~will not work for final exam~~ MIGHT WORK, ASK DR. PROF. LANDON

P56 Let $\mathcal{P} = \text{span}\{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$. We define the affine space $A = x_o + \mathcal{P} = \{x_o + v \mid v \in \mathcal{P}\}$ where $x_o \in \mathbb{R}^n$ is the base-point of A (x_o is not unique)
Show $T : \mathbb{R}^n \to \mathbb{R}^m$ linear $\Rightarrow$ $T(A)$ is affine space.

Consider,

$$\begin{aligned}
T(A) &= \{T(x) \mid x \in A\} && \text{: def}^{\underline{n}} \text{ of image of } A \text{ under } T. \\
&= \{T(x_o+v) \mid v \in \mathcal{P}\} && \text{: def}^{\underline{n}} \text{ of } A = x_o + \mathcal{P}. \\
&= \{T(x_o) + T(v) \mid v \in \mathcal{P}\} && \text{: additivity of } T \\
&= T(x_o) + \underbrace{\{T(v) \mid v \in \text{span}\{v_1, v_2, \dots, v_k\}\}}_{S'}
\end{aligned}$$

It remains to show S' is a span of particular vectors in $\mathbb{R}^m$. Notice $v \in \text{span}\{v_1, v_2, \dots, v_k\}$
$\Rightarrow v = \sum_{i=1}^{k} c_i v_i$ thus $T(v) = T\left(\sum_{i=1}^{k} c_i v_i\right) = \sum_{i=1}^{k} c_i T(v_i)$

which shows $T(v) \in \text{span}\{T(v_1), T(v_2), \dots, T(v_k)\}$. But, this shows $S' \subseteq \text{span}\{T(v_1), T(v_2), \dots, T(v_k)\}$. Conversely, if $x \in \text{span}\{T(v_1), \dots, T(v_k)\} \Rightarrow x = \sum_{i=1}^{k} c_i T(v_i) = T\left(\sum_{i=1}^{k} c_i v_i\right)$

but, $\sum_{i=1}^{k} c_i v_i \in \text{span}\{v_1, \dots, v_k\}$ thus $x = T(v)$ for some $v \in \text{span}\{v_1, \dots, v_k\}$ $\therefore$ $x \in S'$ $\Rightarrow$ $\text{span}\{T(v_1), \dots, T(v_k)\} \subseteq S'$.
We find, $S' = \text{span}\{T(v_1), T(v_2), \dots, T(v_k)\}$ an conclude,

$$T(A) = T(x_o) + \text{span}\{T(v_1), T(v_2), \dots, T(v_k)\}.$$

Hence $T(A)$ is an affine space in $\mathbb{R}^m$ with base point $T(x_o)$.

P57 Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be linear trans. with $T(e_1) = (1,1,2)$, $T(e_2) = (2,2,1)$ and $T(e_3) = (1,0,1)$. Show T is injective and surjective. Also, find f-la for T^{-1}

Consider the following calculation, it will provide the needed features for T after some thought.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 1 & 2 & 0 & b \\ 2 & 1 & 1 & c \end{array}\right] \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 0 & -1 & b-a \\ 0 & -3 & -1 & c-2a \end{array}\right] \xrightarrow[2r_2]{3r_1} \left[\begin{array}{ccc|c} 3 & 6 & 3 & 3a \\ 0 & 0 & -1 & b-a \\ 0 & -6 & -2 & 2c-4a \end{array}\right]$$

$$\xrightarrow{r_1 + r_3} \left[\begin{array}{ccc|c} 3 & 0 & 1 & 2c-a \\ 0 & 0 & -1 & b-a \\ 0 & -6 & -2 & 2c-4a \end{array}\right] \xrightarrow[r_3 - 2r_2]{r_1 + r_2} \left[\begin{array}{ccc|c} 3 & 0 & 0 & -2a+b+2c \\ 0 & 0 & -1 & b-a \\ 0 & -6 & 0 & \underbrace{2c-4a-2(b-a)}_{-2a-2b+2c} \end{array}\right]$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 3 & 0 & 0 & -2a+b+2c \\ 0 & -6 & 0 & -2a-2b+2c \\ 0 & 0 & -1 & b-a \end{array}\right]$$

$$\xrightarrow[-r_3]{\xrightarrow[r_2/-6]{r_1/3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3}(-2a+b+2c) \\ 0 & 1 & 0 & \frac{1}{6}(2a+2b-2c) \\ 0 & 0 & 1 & a-b \end{array}\right]$$

It follows that $\mathrm{rref}[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ hence the columns of $[T]$ are LI thus T is injective. Furthermore, if $(a,b,c) \in \mathbb{R}^3$ then the calculation above reveals

$$T\left(\tfrac{1}{3}(-2a+b+2c),\ \tfrac{1}{6}(2a+2b-2c),\ a-b \right) = (a,b,c)$$

Hence T is a surjection. Finally, to find T^{-1} f-la we could solve $T(x,y,z) = (a,b,c)$ for (x,y,z), but, again the calculation above reveals:

$$\boxed{T^{-1}(a,b,c) = \left(\tfrac{1}{3}(-2a+b+2c),\ \tfrac{1}{6}(2a+2b-2c),\ a-b \right)}$$

- As a check on my work, $[T][T^{-1}] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 1/3 & 1/3 & -1/3 \\ 1 & -1 & 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Remark: $\exists$ other ways to argue for $\boxed{P57}$. In particular, once justified, the formula for T^{-1} allows for easy proofs of onto and 1-1,

$$T(a) = T(b) \implies T^{-1}(T(a)) = T^{-1}(T(b)) \implies a = b \quad \therefore 1-1.$$

$a \in \mathbb{R}^3$ then $T(T^{-1}(a)) = a$ thus T is onto $\mathbb{R}^3$.

Of course, justifying the existence of T^{-1} is a pain.

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CCW CW (See handout from Anton)

$\boxed{P58}$ T rotate around z-axis by θ, S rotate around y-axis by β. Find the standard matrix for $T \circ S$ and $S \circ T$.

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{fixes } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ see } [T]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(axes fixed under rotation in $\mathbb{R}^3$)

$$[S] = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \quad \text{fixes } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

which shows y-axis is the axis of rotation for S.

Anyway, these you could find in my notes or ask me, I'd tell you.
Next,

$$[T \circ S] = [T][S] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} \cos\theta\cos\beta & -\sin\theta & -\cos\theta\sin\beta \\ \sin\theta\cos\beta & \cos\theta & -\sin\theta\sin\beta \\ \sin\beta & 0 & \cos\beta \end{bmatrix} = [T \circ S]}$$

Likewise,

$$[S \circ T] = [S][T] = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} \cos\beta\cos\theta & -\cos\beta\sin\theta & -\sin\beta \\ \sin\theta & \cos\theta & 0 \\ \sin\beta\cos\theta & -\sin\beta\sin\theta & \cos\beta \end{bmatrix} = [S \circ T]}$$

not always same

rotations in 3D do not generally commute.

Remark: it is possible to show $T \circ S$ and $S \circ T$ are both rotations. However, they do not have the same axis of rotation. What they do share in common is the angle by which they rotate. These things we will eventually learn how to derive. For now, I leave you with this difficult to check remark -(it's easy to check later)-

$\boxed{P\,59}$ Let $T(v) = Av$ define $T : \mathbb{R}^n \to \mathbb{R}^n$ for which the matrix $A \in \mathbb{R}^{n \times n}$ is symmetric; $A^T = A$. Determine how $v \cdot T(w)$ and $T(v) \cdot w$ are related

(Notice $v \cdot T(w)$ is a 1×1 matrix hence $(v \cdot T(w))^T = v \cdot T(w)$) didn't need it.

Moreover,

$$v \cdot T(w) = v^T T(w) = v^T A w$$

$$T(v) \cdot w = (T(v))^T w = (Av)^T w = v^T A^T w$$

But, $A^T = A$ hence $v \cdot T(w) = v^T A w = v^T A^T w = T(v) \cdot w$.

Therefore, $\boxed{v \cdot T(w) = T(v) \cdot w}$ for T induced from a symmetric matrix A.

$\boxed{P\,60}$

$$\left[\frac{d}{dt}(AB)\right]_{ij} = \frac{d}{dt}(AB)_{ij} = \frac{d}{dt}\left(\sum_{k=1}^{p} A_{ik} B_{kj}\right) = \sum_{k=1}^{p} \frac{d}{dt}\left(A_{ik} B_{kj}\right) \quad : \text{linearity of } \frac{d}{dt}.$$

$$= \sum_{k=1}^{p}\left(\frac{dA_{ik}}{dt} B_{kj} + A_{ik}\frac{dB_{kj}}{dt}\right) \quad : \text{product rule of calculus I.}$$

$$= \sum_{k=1}^{p}\left(\frac{dA}{dt}\right)_{ik} B_{kj} + \sum_{k=1}^{p} A_{ik}\left(\frac{dB}{dt}\right)_{kj} \quad : \text{def}^{\underline{n}} \text{ of } \tfrac{d}{dt} \text{ of matrix}$$

$$= \left(\frac{dA}{dt}B + A\frac{dB}{dt}\right)_{ij} \quad : \text{def}^{\underline{n}} \text{ of matrix mult. and addition.}$$

Hence, $\boxed{\frac{d}{dt}(AB) = \frac{dA}{dt}B + A\frac{dB}{dt}}$ as the above holds $\forall i,j$.