HOMEWORK 32

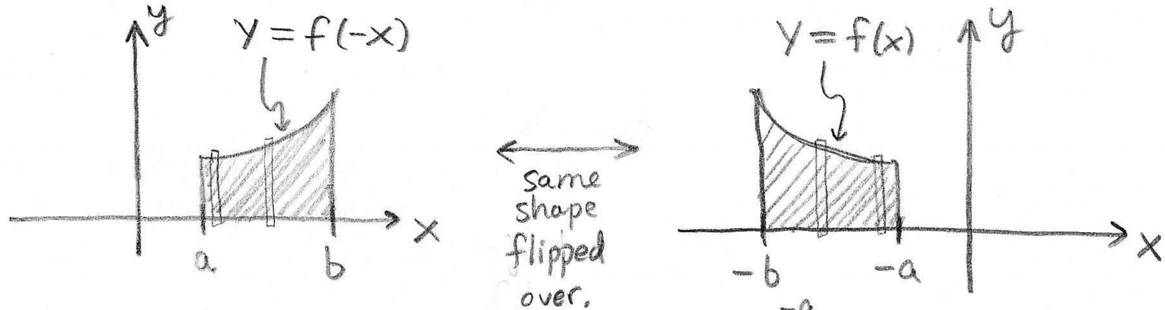
§5.5 #59] Suppose f is continuous on $\mathbb{R}$. Show $\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$.

$$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du)$$: letting $u = -x$ so $du = -dx$ and $u(a) = -a$, $u(b) = -b$.

$$= -\int_{-b}^{-a} f(u)(-du)$$

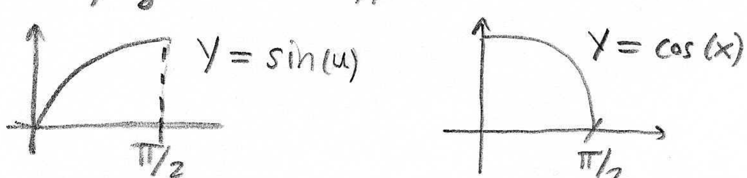
$$= \int_{-b}^{-a} f(x)\,dx,$$: switching dummy variable of integration from "u" to "x".

The graphical meaning of this in the case $f(x) \geq 0$ and $0 < a < b$ is as follows,



$\int_a^b f(-x)\,dx$ $\qquad$ $\int_{-b}^{-a} f(x)\,dx$

§5.5#63] Show that $\int_0^{\pi/2} f(\cos(x))\,dx = \int_0^{\pi/2} f(\sin(x))\,dx$. Notice this is based on same idea as #59. The inputs to f in both cases are same, just in opposite order.



Hmm... notice
$\sin(u) = \cos(\frac{\pi}{2} - u)$
$= \cos(x)$
$\Rightarrow x = \frac{\pi}{2} - u$
$u = \frac{\pi}{2} - x$

Let's show $\int_0^{\pi/2} f(\cos(x))\,dx = \int_0^{\pi/2} f(\sin(u))\,du$.

$$\int_0^{\pi/2} f(\cos(x))\,dx = \int_{\pi/2}^{0} f\left(\cos\left(\tfrac{\pi}{2} - u\right)\right)(-du)$$ $\boxed{u = \frac{\pi}{2} - x,\ du = -dx,\ u(0) = \frac{\pi}{2},\ u(\frac{\pi}{2}) = 0}$

$$= \int_0^{\pi/2} f\left(\cos\left(\tfrac{\pi}{2}\right)\cos(-u) - \sin\left(\tfrac{\pi}{2}\right)\sin(-u)\right)du$$

$$= \int_0^{\pi/2} f(\sin(u))\,du$$

$$= \int_0^{\pi/2} f(\sin(x))\,dx.$$

Remark: this problem req^{d} some thinking.

§5.5 #64 | Problem #63 shows $\int_0^{\pi/2} f(\cos(x))\,dx = \int_0^{\pi/2} f(\sin(x))\,dx$

Suppose $f(u) = u^2$ then $f(\cos(x)) = \cos^2(x)$ & $f(\sin(x)) = \sin^2(x)$.

$$\int_0^{\pi/2} \cos^2(x)\,dx = \int_0^{\pi/2} f(\cos(x))\,dx = \int_0^{\pi/2} f(\sin(x))\,dx = \int_0^{\pi/2} \sin^2(x)\,dx \equiv I$$

But, notice $\cos^2(x) + \sin^2(x) = 1$

$$\underbrace{\int_0^{\pi/2} (\cos^2(x) + \sin^2(x))\,dx}_{\displaystyle \int_0^{\pi/2} \cos^2(x)\,dx + \int_0^{\pi/2} \sin^2(x)\,dx} = \int_0^{\pi/2} 1\,dx = x\Big|_0^{\pi/2} = \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^2(x)\,dx + \int_0^{\pi/2} \sin^2(x)\,dx = I + I = 2I = \frac{\pi}{2} \Rightarrow \underline{I = \frac{\pi}{4}}$$

$$\therefore \boxed{\int_0^{\pi/2} \cos^2(x)\,dx = \int_0^{\pi/2} \sin^2(x)\,dx = \frac{\pi}{4}}$$

Remark: this is cute and all but the simple trig. identities

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ & $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

provide nicer sol$^{\underline{n}}$'s via u-substitution.

$$\begin{aligned}
\int_0^{\pi/2} \cos^2(x)\,dx &= \int_0^{\pi/2} \frac{1}{2}(1 + \cos(2x))\,dx \\
&= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)dx \\
&= \left(\frac{x}{2} + \frac{1}{4}\sin(2x)\right)\Big|_0^{\pi/2} \\
&= \left(\frac{\pi}{4} + \frac{1}{4}\sin(\pi)\right) - \left(0 + \frac{1}{4}\sin(0)\right) \\
&= \boxed{\frac{\pi}{4}}
\end{aligned}$$

I prefer this direct argument. The ideas of #64 & #63 are certainly novel.

§5.5#65

$$\int \frac{dx}{5-3x} = \int \frac{-1}{3u}\,du \longleftarrow \boxed{\begin{aligned} u &= 5-3x \\ du &= -3dx \end{aligned}}$$

$$= \frac{-1}{3}\ln|u| + C$$

$$= \boxed{\frac{-1}{3}\ln|5-3x| + C}$$

§5.5#67

$$\int (\ln(x))^2 \frac{dx}{x} = \int (\ln(x))^2 d(\ln(x)) \longleftarrow \boxed{\begin{aligned} \frac{d(\ln(x))}{dx} &= \frac{1}{x} \\ d(\ln(x)) &= \frac{dx}{x} \end{aligned}}$$

$$= \boxed{\frac{1}{3}(\ln(x))^3 + C}$$

§5.5#69

$$\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du \longleftarrow \boxed{\begin{aligned} \text{let } u &= 1+e^x \\ du &= e^x dx \end{aligned}}$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{2}{3}(1+e^x)^{3/2} + C}$$

§5.5#71

$$\int e^{\tan(x)}\sec^2(x)\,dx = \int e^u\,du \longleftarrow \boxed{\begin{aligned} u &= \tan(x) \\ du &= \sec^2(x)\,dx \end{aligned}}$$

$$= e^u + C$$

$$= \boxed{e^{\tan(x)} + C}$$

§5.5#73

$$\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$$

$$= \tan^{-1}(x) + \int \frac{1}{2u}\,du \longleftarrow \boxed{\begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \end{aligned}}$$

$$= \boxed{\tan^{-1}(x) + \frac{1}{2}\ln|1+x^2| + C}$$

§5.5 #75

$$\int \frac{\sin(2x)}{1+\cos^2(x)}\,dx = \int \frac{du}{u}$$

Let $u = 1+\cos^2(x)$
$du = -2\cos(x)\sin(x)dx$
$\Rightarrow \underline{-du = \sin(2x)dx}$.

$$= \ln|u| + c$$
$$= \boxed{\ln|1+\cos^2(x)| + C}$$

<u>Remark</u>: Recall $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$ & $\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta})$

Thus
$$2\sin\theta\cos\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta})(e^{i\theta}+e^{-i\theta})$$
$$= \frac{1}{2i}(e^{2i\theta}+e^{0}-e^{0}-e^{-2i\theta})$$
$$= \frac{1}{2i}(e^{2i\theta}-e^{-2i\theta})$$
$$= \sin(2\theta) \quad \therefore \quad \boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

We can <u>derive</u> trig. identities via $e^{i\theta}$ & $e^{-i\theta}$.

§5.5 #77

$$\int \cot(x)\,dx = \int \frac{\cos(x)dx}{\sin(x)} = \int \frac{d(\sin(x))}{\sin(x)} = \boxed{\ln|\sin(x)| + C}$$

§5.5 #79

$$\int_e^{e^4} \frac{1}{\sqrt{x}}\frac{dx}{x} = \int_1^4 \frac{du}{\sqrt{u}}$$

$u = \ln(x)$ $u(e^4) = \ln(e^4) = 4$.
$du = \frac{dx}{x}$ $u(e) = \ln(e) = 1$.

$$= 2\sqrt{u}\Big|_1^4$$
$$= 2(\sqrt{4}-\sqrt{1})$$
$$= \boxed{2}$$

§5.5 #81

$$\int_0^1 \frac{e^z+1}{e^z+z}\,dz = \int_1^{e+1}\frac{du}{u}$$

$u = e^z + z$ $u(1) = e+1$
$du = (e^z+1)dz$ $u(0) = 1$

$$= \ln|u|\Big|_1^{e+1}$$
$$= \ln(e+1) - \ln(1)$$
$$= \boxed{\ln(e+1)}$$