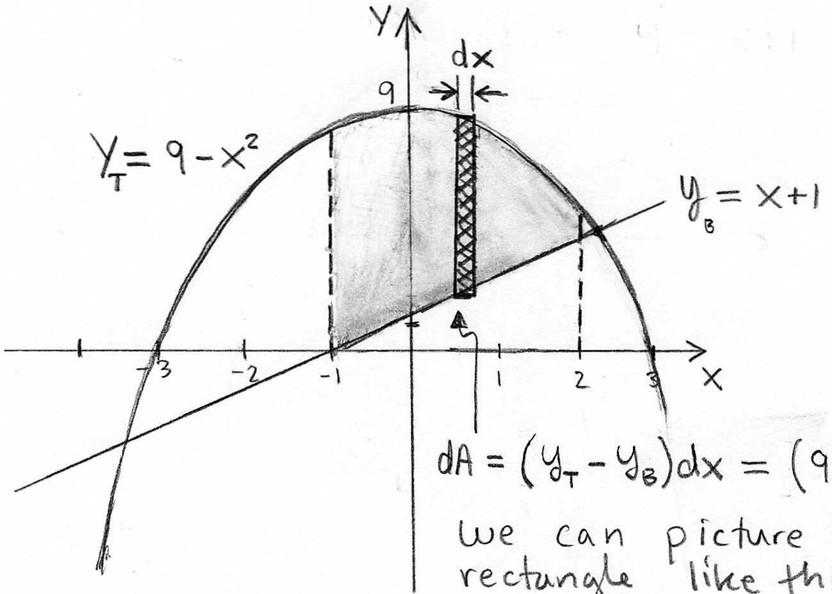
Homework 37, Calculus I

§6.1#5 Find area bounded by $y = x+1$, $y = 9 - x^2$, $x = -1$, $x = 2$



intersection of $y_B = x+1$ and $y_T = 9 - x^2$ has

$$9 - x^2 = x + 1$$
$$x^2 + x - 7 = 0$$
$$x = \frac{-1 \pm \sqrt{1+28}}{2}$$
$$x = \frac{-1 \pm \sqrt{29}}{2} = 2.193 \text{ or } -3.193$$

$$dA = (y_T - y_B)dx = (9 - x^2 - (x+1))\,dx$$

we can picture an infinitesimal rectangle like this for each $x \in [-1, 2]$.

$$A = \int_{-1}^{2} (8 - x^2 - x)\,dx$$
$$= \left(8x - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right|_{-1}^{2}$$
$$= \left(16 - \tfrac{8}{3} - 2\right) - \left(-8 + \tfrac{1}{3} - \tfrac{1}{2}\right)$$
$$= 34/3 + 49/6$$
$$= \boxed{39/2}$$

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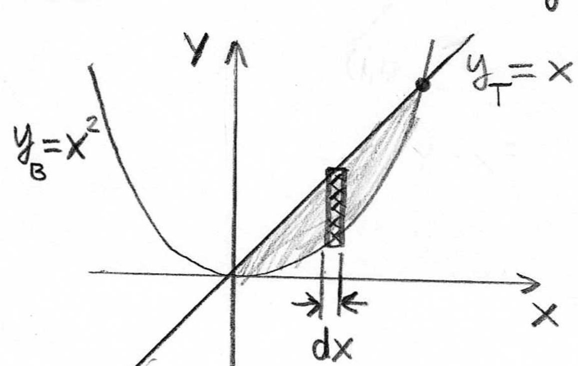
§6.1#7 Find area bounded by $y = x$ and $y = x^2$



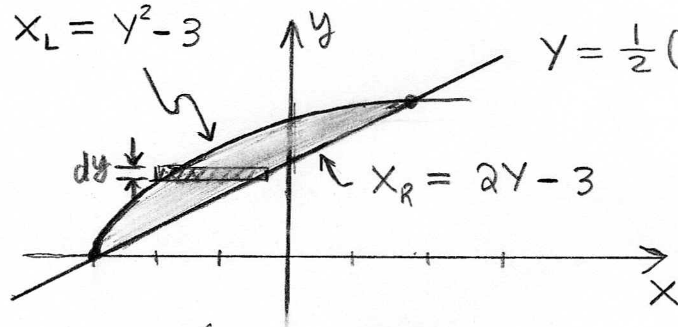
intersection where $x = x^2$ which says $x(x-1) = 0$
$x = 0$ or $x = 1$.

$$dA = (x - x^2)dx$$
(for each $x \in [0, 1]$)

$$A = \int_0^1 (x - x^2)\,dx$$
$$= \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right|_0^1$$
$$= \frac{1}{2} - \frac{1}{3}$$
$$= \boxed{\frac{1}{6}}$$

§6.1#9) find area bounded by $Y=\sqrt{X+3}$ and $Y=\frac{1}{2}(X+3)$

Notice $Y=\sqrt{X+3}$ is top-half of $Y^2=X+3 \rightarrow \underbrace{X=Y^2-3}_{\text{sideways parabola vertex }(-3,0).}$



Also note $y=\frac{1}{2}(X+3)$ has X-intercept $X=-3$.
and y-intercept $y=3/2=1.5$
$Y=\sqrt{X+3}$ has y-intercept $Y=\sqrt{0+3}=\sqrt{3}\cong 1.732$

$$\begin{aligned} dA &= (X_R-X_L)\,dy \\ &= (2Y-3-(Y^2-3))\,dy \\ &= (2Y-Y^2)\,dy \end{aligned}$$

(can see a rectangle for each y between $y=0$ & $y=\underline{b}$)
need to find this.

Intersection at $X_L=X_R$

$$Y^2-3=2Y-3$$
$$Y^2-2Y=Y(Y-2)=0$$
gives $Y=0$ & $Y=2$, apparently $b=2$.

$$A=\int_0^2(2Y-Y^2)\,dY=\left(Y^2-\frac{1}{3}Y^3\right|_0^2=4-\frac{8}{3}=\boxed{\frac{4}{3}}$$

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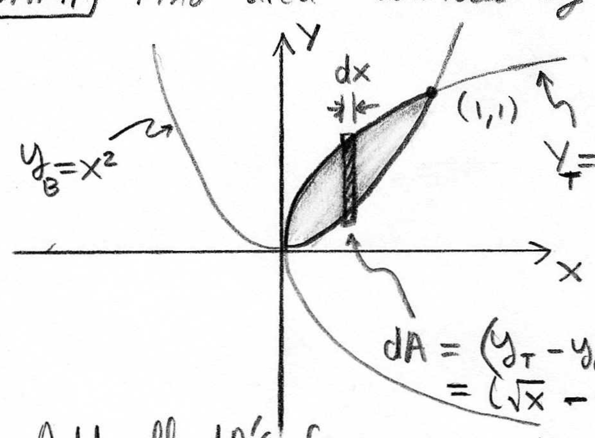
§6.1#11) Find area bounded by $Y=X^2$ and $Y^2=X$



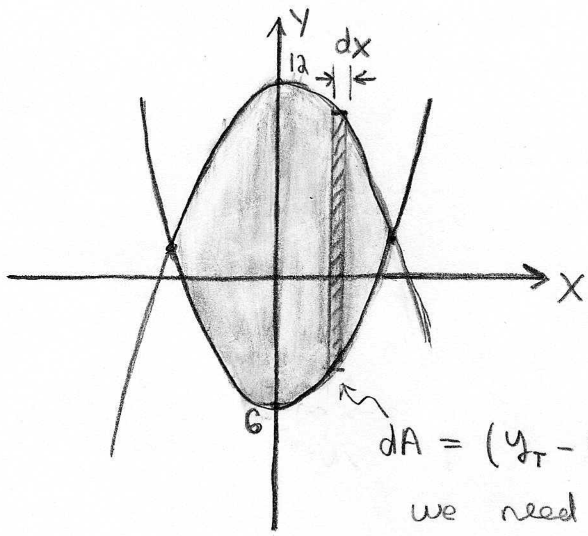
Intersection points have $y_T=y_B$
$$X^2=\sqrt{X}$$
$$X^4=X$$
$$X(X^3-1)=0$$
$\underline{X=0 \text{ or } X=1}$.

$$\begin{aligned} dA &= (y_T-y_B)\,dx \\ &= (\sqrt{x}-x^2)\,dx \end{aligned}$$

Add all dA's from $x=0$ to $x=1$,

$$A=\int_0^1(\sqrt{x}-x^2)\,dx=\left(\frac{2}{3}x^{3/2}-\frac{1}{3}x^3\right|_0^1=\boxed{\frac{1}{3}}$$

§6.1 #13 Find area bounded by $y = 12 - x^2$ and $y = x^2 - 6$.



Clearly $y_T = 12 - x^2$, $y_B = x^2 - 6$

Intersection:

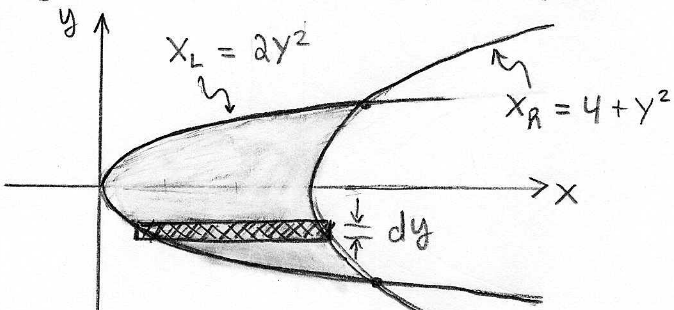
$$12 - x^2 = x^2 - 6$$
$$2x^2 = 18$$
$$x^2 = 9$$
$$\underline{x = \pm 3}$$

$$dA = (y_T - y_B)\,dx = (12 - x^2 - (x^2 - 6))\,dx = (18 - 2x^2)\,dx$$

we need to add each such dA from $x = -3$ to $x = 3$.

$$A = \int_{-3}^{3} (18 - 2x^2)\,dx = \left(18x - \tfrac{2}{3}x^3\right|_{-3}^{3}$$
$$= \left(18(3) - \tfrac{2}{3}(27)\right) - \left(-3(18) - \tfrac{2}{3}(-27)\right)$$
$$= 3(18) - 18 + 3(18) - 18$$
$$= 4(18)$$
$$= \boxed{72}$$

§6.1 #19 Find area bounded by $x_L = 2y^2$ and $x_R = 4 + y^2$.



Intersections at $x_L = x_R$

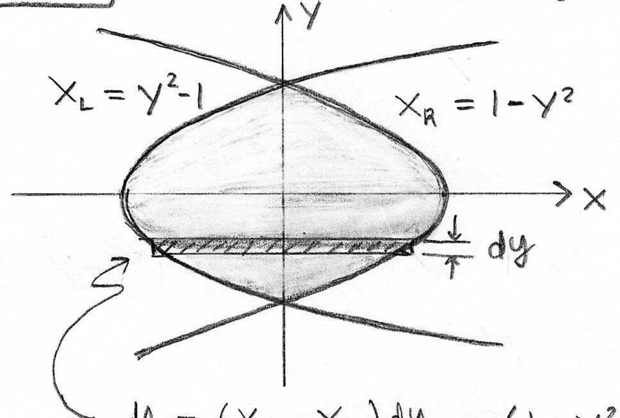
$$2y^2 = 4 + y^2$$
$$y^2 = 4 \therefore \underline{y = \pm 2}$$

So we have a rectangle as pictured for $y \in [-2, 2]$.

$$dA = (x_R - x_L)\,dy = (4 + y^2 - 2y^2)\,dy = (4 - y^2)\,dy$$

$$A = \int_{-2}^{2} (4 - y^2)\,dy$$
$$= \left(4y - \tfrac{1}{3}y^3\right|_{-2}^{2}$$
$$= (8 - 8/3) - (-8 + 8/3)$$
$$= 16 - 16/3$$
$$= \frac{48 - 16}{3}$$
$$= \boxed{32/3}$$

§6.1 #21 Find area bounded by $x = 1 - y^2$ and $x = y^2 - 1$



Intersections at $X_L = X_R$

$$1 - Y^2 = Y^2 - 1$$
$$2Y^2 = 2$$
$$Y = \pm 1$$

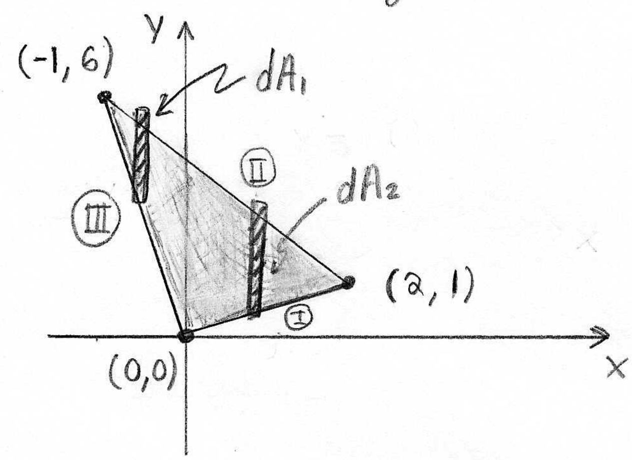
The curves intersect as they cross the y-axis.

$$dA = (X_R - X_L)\,dy = (1 - Y^2 - (Y^2 - 1))\,dy = (2 - 2Y^2)\,dy$$

Add the area of each infinitesimal rectangle from $y = -1$ up to $y = 1$,

$$A = \int_{-1}^{1} 2(1 - Y^2)\,dy = 2\left(y - \tfrac{1}{3}y^3\right)\Big|_{-1}^{1}$$
$$= 2\left(1 - \tfrac{1}{3}\right) - 2\left(-1 + \tfrac{1}{3}\right)$$
$$= \boxed{8/3}$$

§6.1 #29 Find area of triangle with vertices $(0,0)$, $(2,1)$, $(-1,6)$. We'll need to break it into two parts. We need the eq^{n}'s of the sides



(I) Slope $1/2$, $\Rightarrow$ $y = \frac{1}{2}x$

(II) Slope $\frac{1-6}{2+1} = \frac{-5}{3}$

$$y = 6 - \tfrac{5}{3}(x+1) = \tfrac{13}{3} - \tfrac{5}{3}x$$

(III) Is $y = -6x$.

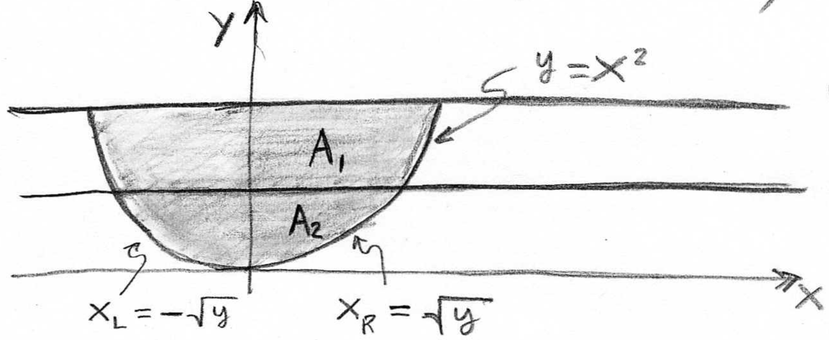
$$dA_1 = (y_{II} - y_{III})\,dx : -1 \le x \le 0$$
$$dA_2 = (y_{II} - y_{I})\,dx : 0 \le x \le 2$$

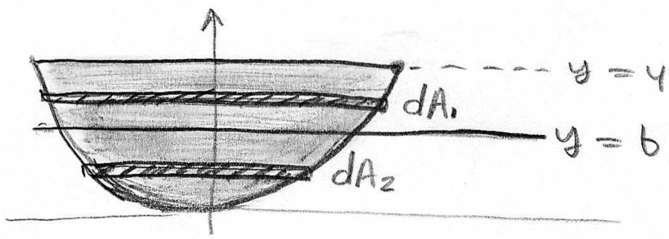
Now add up areas on $[-1, 0]$ and $[0, 2]$. Note $y_{II} - y_{III} = \frac{13}{3} - \frac{5}{3}x + 6x = \frac{13}{3}(1 + x)$

Whereas $y_{II} - y_{I} = \frac{13}{3} - \frac{5}{3}x - \frac{1}{2}x = \frac{13}{3} - \frac{13}{6}x = \frac{13}{6}(2 - x)$

$$A = \int_{-1}^{0} \tfrac{13}{3}(1 + x)\,dx + \int_{0}^{2} \tfrac{13}{6}(2 - x)\,dx$$
$$= \tfrac{13}{3}\left(x + \tfrac{1}{2}x^2\right)\Big|_{-1}^{0} + \tfrac{13}{6}\left(2x - \tfrac{1}{2}x^2\right)\Big|_{0}^{2}$$
$$= \tfrac{13}{3}\left(1 - \tfrac{1}{2}\right) + \tfrac{13}{6}(2) = \tfrac{13}{6} + \tfrac{13}{3} = \frac{13 + 26}{6} = \frac{39}{6} = \boxed{\tfrac{13}{2}}$$

§6.1 # 49 | Find b such that $y = b$ makes regions pictured below have equal area,





$$dA_1 = (x_R - x_L)\,dy = 2\sqrt{y}\,dy \;:\; 0 \le y \le b$$

$$dA_2 = (x_R - x_L)\,dy = 2\sqrt{y}\,dy \;:\; b \le y \le 4$$

We want $A_1 = A_2$ this means

$$\int_0^b 2\sqrt{y}\,dy = \int_b^4 2\sqrt{y}\,dy$$

$$\frac{4}{3} y^{3/2} \Big|_0^b = \frac{4}{3} y^{3/2} \Big|_b^4$$

$$b^{3/2} = 4^{3/2} - b^{3/2}$$

$$2b^{3/2} = 4^{3/2} = \sqrt{64} = 8$$

$$b^{3/2} = 4$$

$$b = 4^{2/3} = \sqrt[3]{16}$$

choose $\boxed{b = \sqrt[3]{16}}$