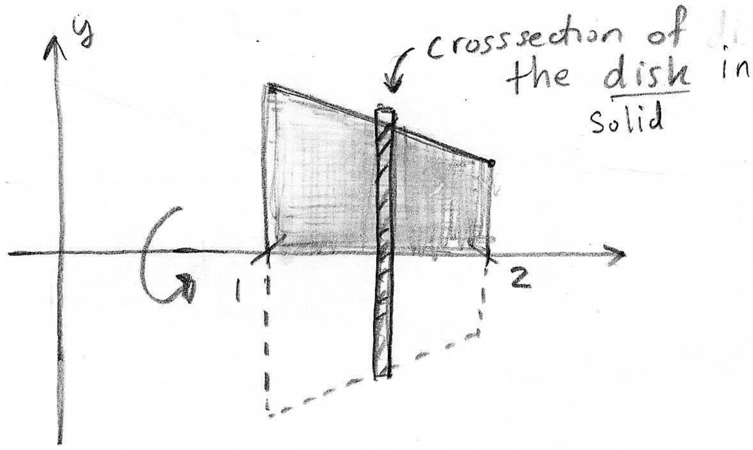
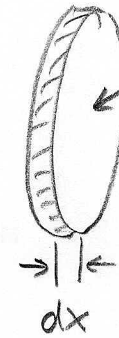
Homework 38, Calculus I

§6.2 #1/ Find volume of solid formed by revolving the area bounded by $y = -\frac{1}{2}x + 2$, $y=0$, $X=1$, $X=2$. around X-axis





$$A = \pi\left(2 - \frac{x}{2}\right)^2 = \pi\left(\tfrac{1}{2}(4-x)\right)^2 = \frac{\pi}{4}(4-x)^2$$

We picture a disk with $dV = \frac{\pi}{4}(4-x)^2 dx$ for each x in $1 \le x \le 2$

$$V = \int_1^2 \frac{\pi}{4}(4-x)^2 dx$$

$$= \frac{\pi}{4}\int_1^2 (16 - 8x + x^2)\,dx$$

$$= \frac{\pi}{4}\left(16x - 4x^2 + \frac{1}{3}x^3\right|_1^2$$

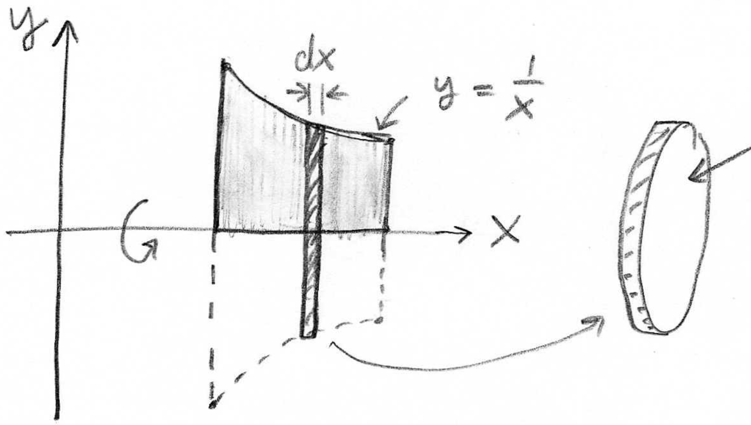
$$= \frac{\pi}{4}\left[\left(32 - 16 + \frac{8}{3}\right) - \left(16 - 4 + \frac{1}{3}\right)\right]$$

$$= \frac{\pi}{4}\left[\frac{48}{3} + \frac{8}{3} - \frac{36}{3} - \frac{1}{3}\right]$$

$$= \frac{19\pi}{12}$$

§6.2#3 | Find volume of solid formed by rotating area bounded by $y = 1/x$, $x=1$, $x=2$, $y=0$ about x-axis.

(2)



$A = \pi y^2$
$= \pi / x^2$

$dV = A\,dx = \frac{\pi}{x^2}\,dx$
for each $x \in [1, 2]$

$$V = \int_1^2 \frac{\pi}{x^2}\,dx = \frac{-\pi}{x}\Big|_1^2 = \pi\left(1 - \tfrac{1}{2}\right) = \boxed{\frac{\pi}{2}}$$

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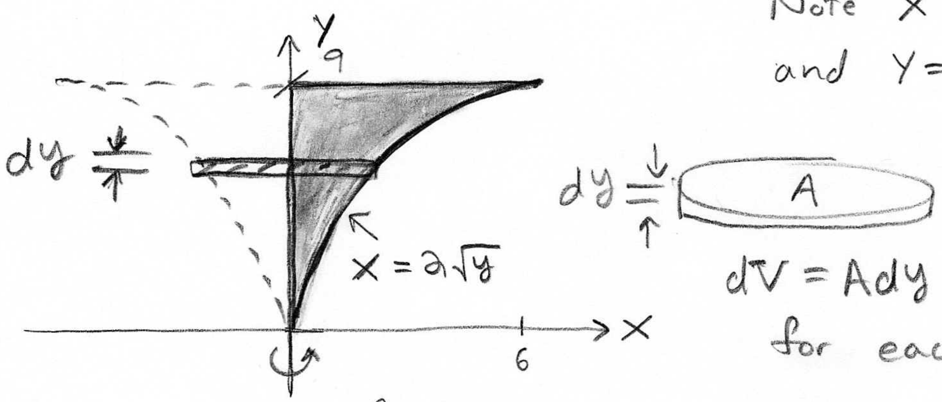
§ 6.2#5 | Find vol. of solid formed by rotating area bounded by $x = 2\sqrt{y}$, $x=0$, $Y=9$ about y-axis.

Note $x = 2\sqrt{y} \rightarrow x^2 = 4y$
and $Y = 9 \Rightarrow x = 2\sqrt{9} = 6.$



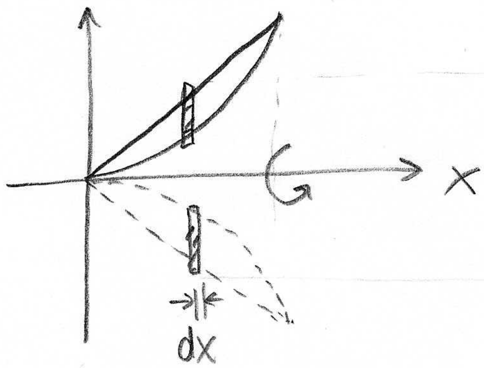
$A = \pi x^2 = 4\pi y$

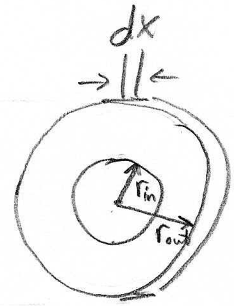
$dV = A\,dy = 4\pi y\,dy$
for each $y \in [0, 9]$

$$V = \int_0^9 4\pi y\,dy = 2\pi y^2\Big|_0^9 = \boxed{162\pi}$$

Remark: my pictures show the side view of the solid. Notice that the dotted-lines indicate the portion of the solid formed from the revolution. Also we see that the axis of rotation suggests the integration be along that area. "Around x" $\rightarrow$ thickness dx $\rightarrow V = \int \_\_\, dx$. whereas "around y" $\longrightarrow$ thickness dy $\longrightarrow V = \int \_\_\, dy$.
You must draw these pictures, although you are free to modify my format in terms of shading and dotted lines etc... Your picture should organize needed info.

§6.2#7 Find volume of solid formed by revolving area bounded by $y = x^3$, $y = x$, $x \geq 0$ around x-axis





$r_{in} = x^3$

$r_{out} = x$

$A = \pi x^2 - \pi x^6$

$dV = \pi(x^2 - x^6)\,dx$

for $x \in [0,1]$.

Notice $x^3 = x \rightarrow x(x^2-1) = 0 \therefore x = 0, \pm 1$. Thus the intersection of $y = x$ and $y = x^3$ is $(0,0)$ and $(1,1)$ for $x \geq 0$. We sum the volume dV of each <u>washer</u> for $0 \leq x \leq 1$,

$$V = \int_0^1 \pi(x^2 - x^6)\,dx = \pi\left(\frac{x^3}{3} - \frac{x^7}{7}\right)\Big|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{7}\right) = \boxed{\frac{4\pi}{21}}$$

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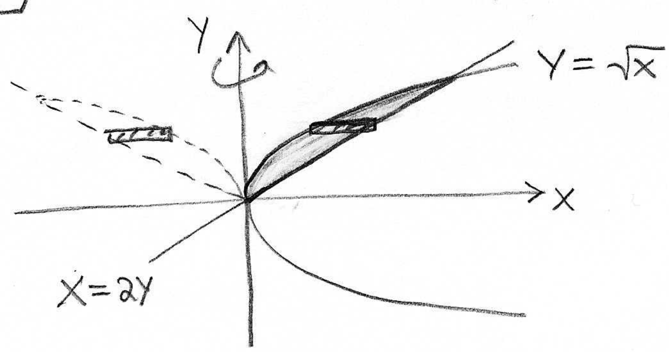
§6.2#9 Find volume obtained by rotating area bounded by $y^2 = x$ and $x = 2y$ around y-axis



$A = \pi r_{out}^2 - \pi r_{in}^2$

$r_{out} = x_R = 2y$

$r_{in} = x_L = y^2$

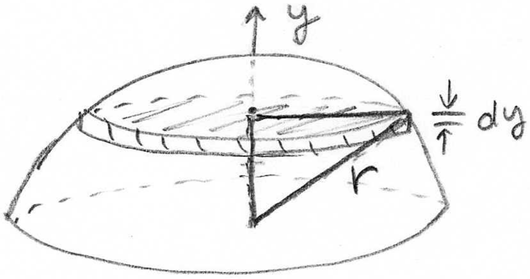
$dV = \pi(4y^2 - y^2)\,dy = 3\pi y^2\,dy$

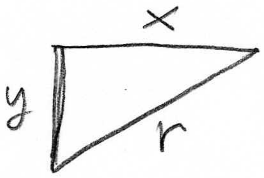
Notice intersection has $y^2 = 2y \Rightarrow y(y-2) = 0 \Rightarrow y = 0$ and $y = 2$ thus we want to add up all the dV's from $y = 0$ to $y = 2$,

$$V = \int_0^2 3\pi y^2\,dy = \pi y^3\Big|_0^2 = \boxed{8\pi}$$

§6.2 #49 $V = \frac{\pi r^2 h}{3}$ See Example 7.4.2

§6.2 #51 Find volume of cap of sphere of radius r. The cap is top part of sphere, h from top.





$$x^2 + y^2 = r^2$$
$$x^2 = r^2 - y^2$$

$$A = \pi x^2 = \pi(r^2 - y^2)$$
$$dV = \pi(r^2 - y^2)\,dy$$

The cap of sphere has $r - h \leq y \leq r$ thus,

$$
\begin{aligned}
V &= \int_{r-h}^{r} \pi(r^2 - y^2)\,dy \\
&= \pi\left[r^2 h - \tfrac{1}{3}(r)^3 + \tfrac{1}{3}(r-h)^3\right] \\
&= \pi\left[r^2 h - \tfrac{1}{3}r^3 + \tfrac{1}{3}(r^3 - 3r^2 h + 3rh^2 - h^3)\right] \\
&= \pi\left[\cancel{r^2 h} - \cancel{r^2 h} + rh^2 - \tfrac{1}{3}h^3\right] \\
&= \pi\left(rh^2 - \tfrac{1}{3}h^3\right) \\
&= \boxed{\tfrac{1}{3}\pi h^2 (3r - h)}
\end{aligned}
$$

Notice when $h = r$ we get $V = \frac{1}{3}\pi r^2 (2r) = \frac{2}{3}\pi r^3$ which is half the volume of a sphere of radius r. A good check on this problem.

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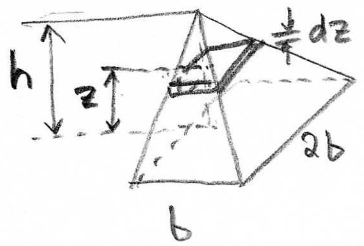
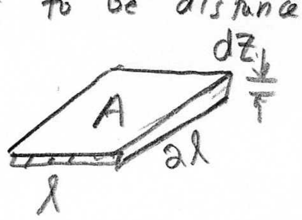
§6.2 #53 We use z to be distance from base. Note that the short side l is a linear function of z. We know $l(z=0) = b$ while $l(z=h) = 0$ hence it follows $l(z) = b - \frac{b}{h}z$.



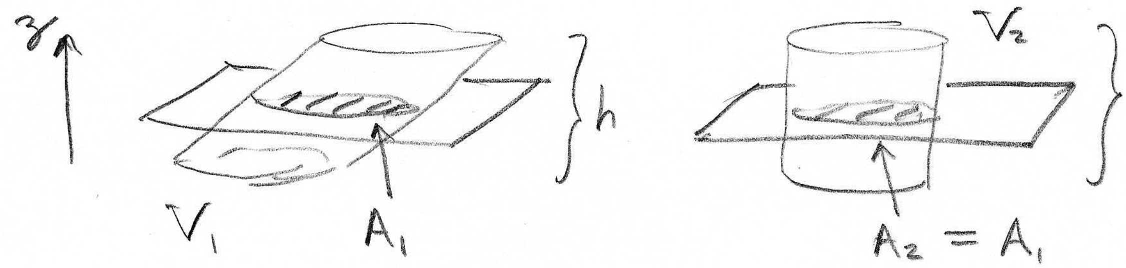


$$A = 2l^2$$
$$dV = 2l^2\,dz$$
$$dV = \frac{2b^2}{h^2}(h - z)^2\,dz$$

$$V = \int_0^h \frac{2b^2}{h^2}(h^2 - 2hz + z^2)\,dz$$
$$= \frac{2b^2}{h^2}\left(h^3 - h^3 + \frac{h^3}{3}\right) = \boxed{\frac{2b^2 h}{3}}$$

Remark: See E13 or E14 for similar problems

§6.2#65 If two solids share equal cross-sectional areas then they have the same volume.



$$V_1 = \int_0^h A_1\,dz \qquad V_2 = \int_0^h A_2\,dz = \int_0^h A_1\,dz = V_1$$

this proves "Cavaliere's Principle".

The volume of the oblique cylinder is the same as that of the straight cylinder, $\boxed{V = \pi r^2 h}$

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§6.2#70 See EIS a.k.a. Example 7.4.15.