November 20

Math 131, Test 4, ~~October 28~~, 2008 Name: ____________________________________.

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. Prepare for math battle. This test has 105 points, 5 are bonus points. Make sure to at least attempt each part.

1. [40pts.] Integrate. For full credit show work.

a.) $\int (x+3)^7 \, dx = \int u^7 \, du$ $\boxed{\begin{array}{l} u = x+3 \\ du = dx \end{array}}$

$$= \frac{1}{8}u^8 + C$$

$$= \boxed{\frac{1}{8}(x+3)^8 + C}$$

b.) $\int \cot(x) \, dx = \int \frac{\cos(x)\,dx}{\sin(x)}$ $\boxed{\begin{array}{l} u = \sin(x) \\ du = \cos(x)\,dx \end{array}}$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\sin(x)| + C}$$

c.) $\displaystyle\int x^2\sqrt{x+1}\,dx = \int (u-1)^2\sqrt{u}\,du \leftarrow \boxed{\begin{array}{c} u = x+1 \\ du = dx \\ x = u-1 \end{array}}$

$= \int \left(u^2\sqrt{u} - 2u\sqrt{u} + \sqrt{u}\right)du$

$= \int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)du$

$= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C}$

d.) $\displaystyle\int \cos^3(2x)\,dx = \int \frac{1}{2}\cos^3(u)\,du \leftarrow \boxed{\begin{array}{c} u = 2x \\ du = 2dx \\ dx = \frac{1}{2}du \end{array}}$

$= \frac{1}{2}\int \cos^2(u)\cos(u)\,du$

$= \frac{1}{2}\int \left(1-\sin^2(u)\right)\cos(u)\,du \rightarrow \boxed{\begin{array}{c} W = \sin(u) \\ dW = \cos(u)du \end{array}}$

$= \frac{1}{2}\int (1 - w^2)\,dw$

$= \frac{1}{2}\left(w - \frac{1}{3}w^3\right) + C$

$= \frac{1}{2}\left(\sin(u) - \frac{1}{3}\sin^3(u)\right) + C$

$= \boxed{\frac{1}{2}\left(\sin(2x) - \frac{1}{3}\sin^3(2x)\right) + C}$

e.) $\displaystyle\int \sqrt{y}\left(\frac{1}{y+y^2}\right) dy = \int \frac{\sqrt{y}\,dy}{y(1+y)}$

$$= \int \left(\frac{1}{1+y}\right)\frac{dy}{\sqrt{y}}$$

$u = \sqrt{y}$

$du = \frac{1}{2\sqrt{y}}\,dy$

$$= \int \left(\frac{1}{1+u^2}\right) 2\,du$$

$$= 2\tan^{-1}(u) + C$$

$$= \boxed{2\tan^{-1}(\sqrt{y}) + C}$$

f.) $\displaystyle\int_2^4 \frac{1}{1-3t}\,dx = \int_2^4 \left(\frac{1}{1-3t}\right) dt$

(typo)

(oops $dx = dt$)

$$= \int_{-5}^{-11} \frac{-\frac{1}{3}\,du}{u}$$

$u = 1-3t$

$u(2) = 1-6 = -5$

$u(4) = 1-12 = -11$

$du = -3\,dt$

$$= -\frac{1}{3}\ln|u| \Big|_{-5}^{-11}$$

$$= -\frac{1}{3}\ln|-11| + \frac{1}{3}\ln|-5|$$

$$= \frac{1}{3}\left(\ln(5) - \ln(11)\right)$$

$$= \boxed{\ln\left(\sqrt[3]{5/11}\right)}$$

2.[10pts] Suppose Dwight has initial position $x(0) = 4$ and initial velocity $v(0) = 8$. If the acceleration of Dwight at time t is $a(t) = t^2 + 1$ then find the velocity $v(t)$ and position $x(t)$ of Dwight.

$$a(t) = \frac{dV}{dt} = t^2 + 1$$

integrate indefinitely w.r.t time t, note $\int \frac{dV}{dt} dt = V + C$

thus

$$V = \frac{1}{3}t^3 + t + C$$

$$V(0) = 8 \Rightarrow 8 = \frac{1}{3}(0^3) + 0 + C \Rightarrow \underline{C = 0}$$

$$\therefore \boxed{V(t) = \frac{1}{3}t^3 + t + 8}$$

Next integrate $V = \frac{dx}{dt}$ to get $\int \frac{dx}{dt} dt = X + C$

thus,

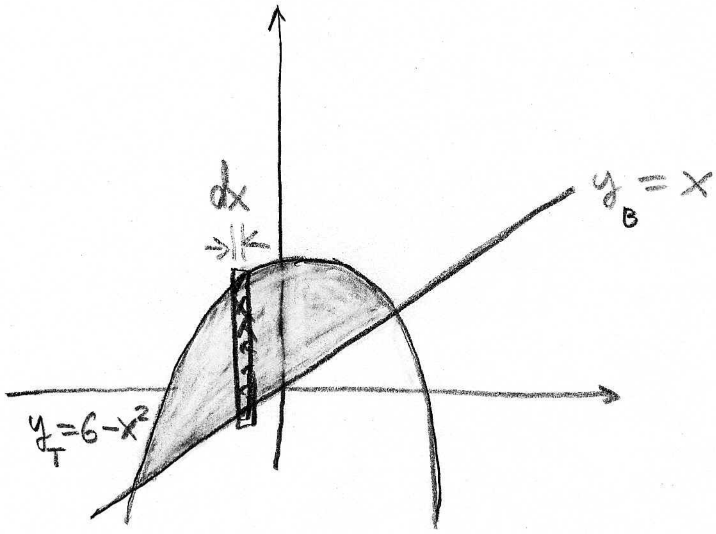
$$X(t) = \frac{1}{12}t^4 + \frac{1}{2}t^2 + 8t + C_2$$

$$X(0) = 4 = \frac{1}{12}(0)^4 + \frac{1}{2}(0)^2 + 8(0) + C_2$$

$$\Rightarrow \underline{C_2 = 4}$$

$$\therefore \boxed{X(t) = \frac{1}{12}t^4 + \frac{1}{2}t^2 + 8t + 4}$$

3.[25pts] Find area bounded by the curves $x = y$ and $y = 6 - x^2$. Your solution should include a graph which indicates how you set up dA.



Intersection Points?

$y_T = y_B$

$6 - x^2 = x$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = 2$ & $x = -3$

$$dA = (y_T - y_B)\,dx$$
$$= (6 - x^2 - x)\,dx \quad \text{for } -3 \le x \le 2$$

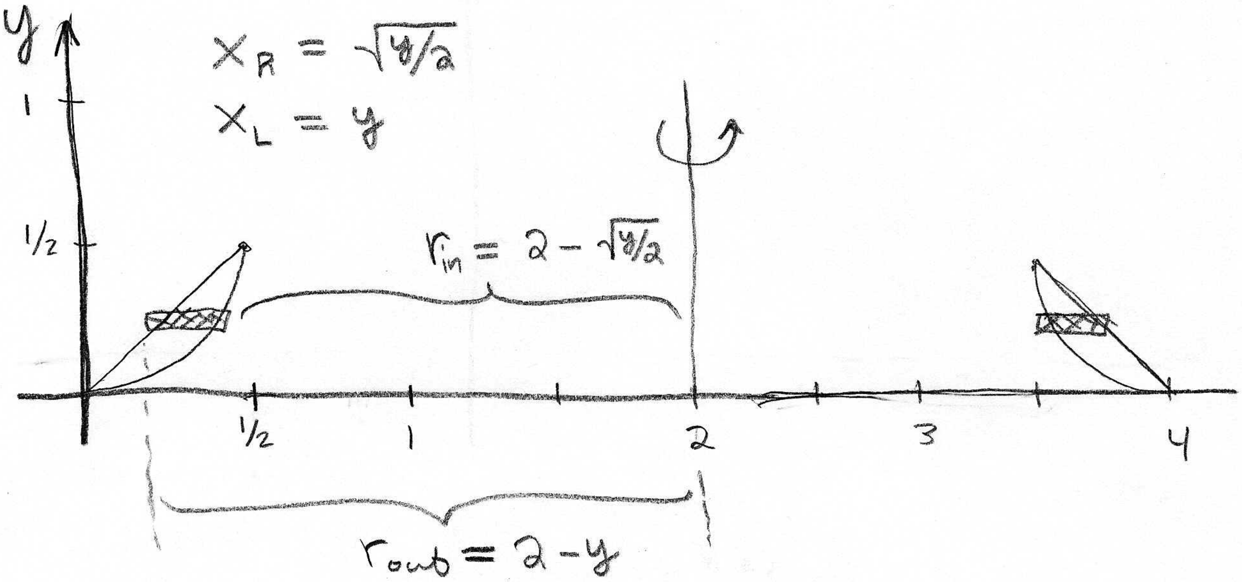
$$A = \int_{-3}^{2} (6 - x^2 - x)\,dx$$
$$= \left(6x - \frac{x^3}{3} - \frac{x^2}{2}\right|_{-3}^{2}$$
$$= \left(6(2) - \frac{8}{3} - \frac{4}{2}\right) - \left(6(-3) + \frac{27}{3} - \frac{9}{2}\right)$$
$$= \left(12 - \frac{8}{3} - 2\right) - \left(-18 + 9 - \frac{9}{2}\right)$$
$$= 10 - \frac{8}{3} + 9 + \frac{9}{2}$$
$$= 19 - \frac{16}{6} + \frac{27}{6}$$
$$= 19 + \frac{11}{6}$$
$$= \boxed{\frac{125}{6}}$$

4.[25pts] Calculate the volume of the solid which is obtained by rotating the area bounded by $y = x$ and $y = 2x^2$ around the $x = 2$ axis.

Intersection Points?

$$x = 2x^2$$
$$2x^2 - x = 0$$
$$(2x-1)x = 0$$
$$x = 0,\ x = 1/2$$

$$x_R = \sqrt{y/2}$$
$$x_L = y$$

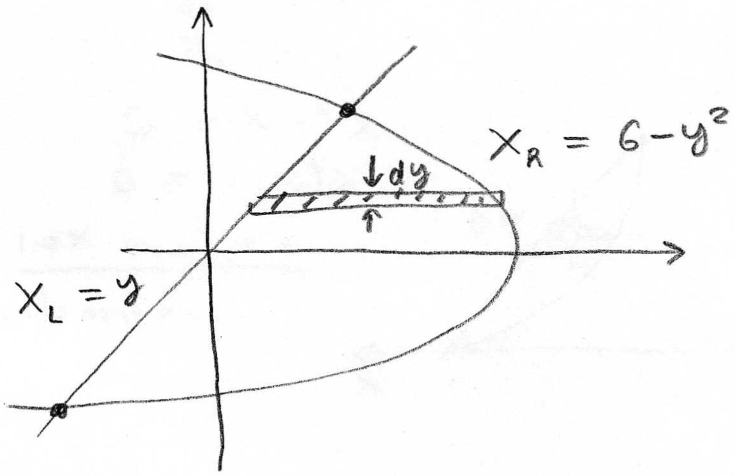


$$dV = \pi\left(r_{out}^2 - r_{in}^2\right)dy$$
$$= \pi\left((2-y)^2 - \left(2 - \sqrt{y/2}\right)^2\right)dy$$
$$= \pi\left(\cancel{4} - 4y + y^2 - \left(\cancel{4} - 4\sqrt{y/2} + y/2\right)\right)dy$$
$$= \pi\left(y^2 - \frac{9}{2}y + 4\sqrt{\frac{y}{2}}\right)dy \ ; \quad 0 \le y \le 1/2$$

Then sum the volume of the infinitesimal washers,

$$V = \int_0^{1/2} \pi\left(y^2 - \frac{9}{2}y + 4\sqrt{\frac{y}{2}}\right)dy = \boxed{\frac{7\pi}{48}}$$

3.[25pts] Find area bounded by the curves $x = y$ and $x = 6 - y^2$. Your solution should include a graph which indicates how you set up dA.



Intersection Points have

$$X_L = X_R$$

$$y = 6 - y^2$$

$$y^2 + y - 6 = 0$$

$$(y+3)(y-2) = 0$$

$$y = -3 \text{ or } y = 2$$

$$dA = (X_R - X_L)\,dy$$

$$= (6 - y^2 - y)\,dy \quad \text{for } -3 \le y \le 2.$$

$$A = \int_{-3}^{2} (6 - y^2 - y)\,dy$$

$$= \left(6y - \frac{1}{3}y^3 - \frac{y^2}{2}\right)\Big|_{-3}^{2}$$

$$= \left(6(2) - \frac{8}{3} - 2\right) - \left(-18 + 9 - \frac{9}{2}\right)$$

$$= \left(10 - \frac{8}{3}\right) - \left(-9 - \frac{9}{2}\right)$$

$$= \frac{22}{3} + \frac{27}{2}$$

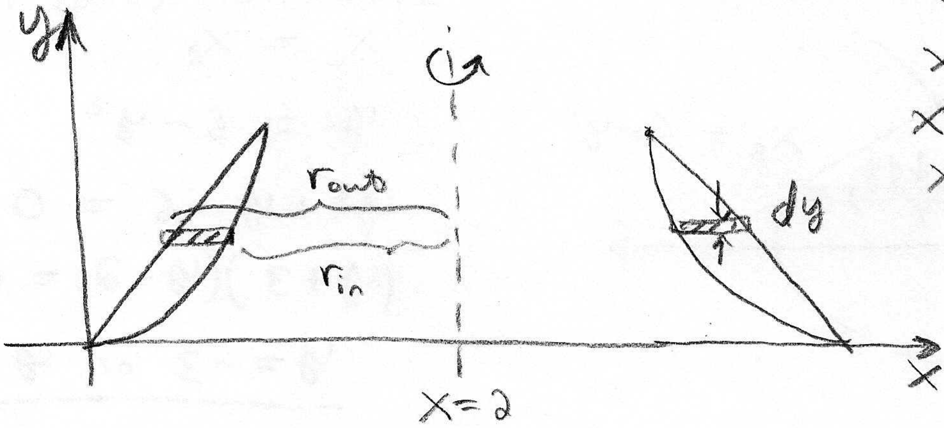
$$= \frac{44 + 81}{6}$$

$$= \boxed{\frac{125}{6}}$$

neat, same as the other version
nice symmetry here.

4.[25pts] Calculate the volume of the solid which is obtained by rotating the area bounded by $y = x$ and $y = x^2$ around the $x = 2$ axis.

$$y_T = y_B$$
$$x = x^2$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$\underline{x = 0 \quad \text{or} \quad x = 1}$$
intersections.

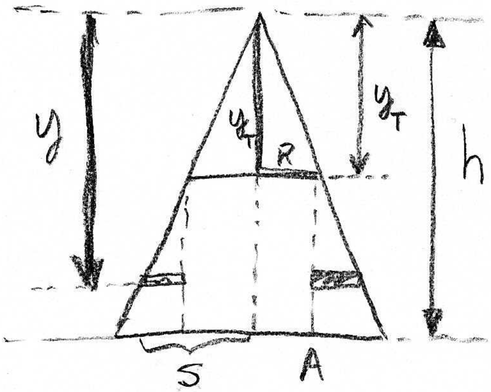


$$r_{out} = 2 - x_L = 2 - y$$
$$r_{in} = 2 - x_R = 2 - \sqrt{y}$$

$$dV = \pi\left(r_{out}^2 - r_{in}^2\right)dy$$
$$= \pi\left((2-y)^2 - (2-\sqrt{y})^2\right)dy$$
$$= \pi\left(4 - 4y + y^2 - 4 + 4\sqrt{y} - y\right)dy$$
$$= \pi\left(y^2 - 5y + 4\sqrt{y}\right) \text{ for } 0 \le y \le 1.$$

$$V = \int_0^1 \pi\left(y^2 - 5y + 4\sqrt{y}\right)dy$$
$$= \pi\left(\frac{1}{3} - \frac{5}{2} + \frac{8}{3}\right)$$
$$= \pi\left(\frac{2 - 15 + 16}{6}\right)$$
$$= \frac{3\pi}{6}$$
$$= \boxed{\frac{\pi}{2}}$$

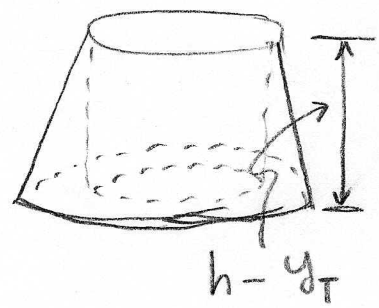
5.[5pts] Calculate the volume of a right circular cone of height h and radius A which has a hole of radius R drilled vertically from the point of the cone all the way to the center of its base.

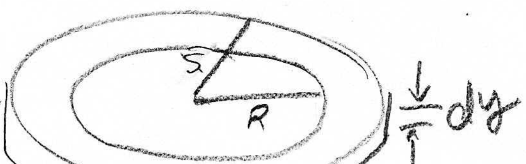


Similar triangles,

$$\frac{y_T}{R} = \frac{h}{A}$$

$$\therefore \quad \underline{y_T = \frac{h}{A} R}$$





$$dV = \pi(s^2 - R^2)\,dy$$

for $y_T \le y \le h$

$$V = \int_{y_T}^{h} \pi(s^2 - R^2)\,dy \qquad \underline{s = \frac{A}{h} y}$$

$$= \int_{y_T}^{h} \pi\left(\frac{A^2}{h^2} y^2 - R^2\right) dy$$

$$= \pi\left(\frac{A^2}{3h^2}(h^3 - y_T^3) - R^2(h - y_T)\right)$$

$$= \pi\left[\frac{A^2}{3h^2}\left(h^3 - \left(\frac{hR}{A}\right)^3\right) - R^2\left(h - \frac{hR}{A}\right)\right]$$

$$= \pi\left(\frac{1}{3}A^2 h - \frac{1}{3} hR^3/A - R^2 h + hR^3/A\right)$$

$$= \boxed{\pi\left(\frac{1}{3}A^2 + \frac{2}{3}\frac{R^3}{A} - R^2\right) h = V}$$

Note $R = 0$ yields $V = \frac{1}{3}\pi A^2 h$ (checks cone case.)