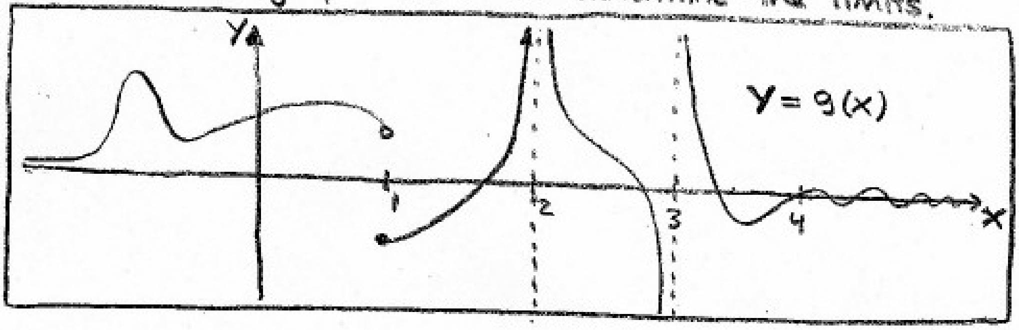
MA 141 - 003, TEST I, SUMMER II 2005

WRITE ON SEPERATE SHEETS KEEP THIS ONE.

1. (15pts.) Let $f(x) = -\sqrt{x-2}$.
   a.) Graph $Y = f(x)$.
   b.) State the domain and range of f.
   c.) Is f one-to-one? Explain.
   d.) Find $f^{-1}(x)$ if possible.
   e.) Graph $Y = f^{-1}(x)$ on the same graph as a.).

2. (3pts.) Carefully define continuity of $f(x)$ at a.

3. (12pts.) Use the graph below to determine the limits.



a.) $\lim_{x \to 4} (g(x))$

b.) $\lim_{x \to 3^+} (g(x))$

c.) $\lim_{x \to 2} (g(x))$

d.) $\lim_{x \to \infty} (g(x))$

4. (20pts.) Use algebra and the ideas from lecture to calculate the limits.

a.) $\lim_{x \to 2} \left(\frac{2x^2+1}{x^2+6x-4} \right)$

b.) $\lim_{x \to 3} \left(\frac{\cos(\pi x)(x-3)}{x^2-5x+6} \right)$

c.) $\lim_{x \to -2} \sqrt{x^4+3x+6}$

d.) $\lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right)$

5. (20pts.) Use your knowledge of elementary functions and perhaps some algebra to calculate,

a.) $\lim_{x \to \infty} \left(\frac{ax^2+bx+c}{x^2-2} \right)$

b.) $\lim_{x \to \infty} \left(3\tan^{-1}(x) \right)$

c.) $\lim_{x \to 0^-} \left(\frac{1}{x} \right)$

d.) $\lim_{x \to 3^+} \left(\ln(x-3) \right)$

6. (3pts.) Define $f'(x)$ in terms of a limiting process.

7. (7pts.) Prove that $\frac{d}{dx}(\sin(x)) = \cos(x)$. Recall that you will need to use the trig. identity $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$. Also recall from lecture that $\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right) = 1$ and $\lim_{h\to 0}\left(\frac{1-\cos(h)}{h}\right) = 0$

8. (20pts.) Calculate the following derivatives (a, b, c are constants)

a.) $\frac{d}{d\theta}\left(\sin(\theta) + 3\right)$

b.) $\frac{d}{dx}\left(x\left(a\sqrt{x} + bx^2\right)\right)$

c.) $\frac{d}{dx}\left(ce^x - \frac{5}{\sqrt{x^3}}\right)$

d.) $\frac{d}{dx}\left(\frac{x^2+3}{\sqrt{x}}\right)$

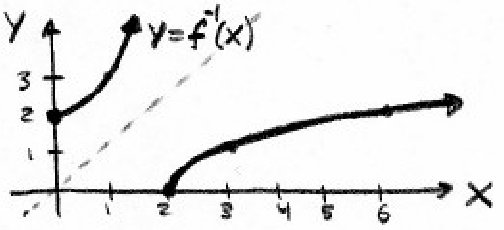
9. (5pts.) Let $f(x) = \sqrt{x}$ find equation of the tangent line to $y = f(x)$ through $(4, 2)$.

BONUS: Prove that if $f(x)$ is differentiable at a then it must be continous at a. (5pts.

MA 141: TEST I SOLUTION

$\boxed{1.}$ $f(x) = \sqrt{x-2}$

a.) The graph is $\sqrt{x}$ shifted 2 units right



b.) $\text{dom}(f) = [2,\infty)$ and $\text{range}(f) = [0,\infty)$.

c.) f is one-to-one because it passes the horizontal line test. (Alternatively $\sqrt{a-2} = \sqrt{b-2} \Rightarrow a-2 = b-2 \Rightarrow a = b$ thus $f(a) = f(b) \Longrightarrow a = b$ which proves f is 1-1.)

d.) $y = \sqrt{x-2}$: begin with $f(x) = y$

$x = \sqrt{y-2}$: switch $x \leftrightarrow y$

$x^2 = y-2$: squaring both sides

$y = x^2+2$: solving for y

$\boxed{f^{-1}(x) = x^2+2}$ (for $x \geq 0$ because $\text{dom}(f^{-1}) = \text{range}(f)$)

(2.) f is continous at a if $\lim_{x\to a}(f(x)) = f(a)$

(3.) a.) $\lim_{x\to 4} g(x) = 0$

b.) $\lim_{x\to 3^+} g(x) = \infty$

c.) $\lim_{x\to 2} g(x) = \infty$

d.) $\lim_{x\to\infty} g(x) = 0$

(4.) a.) $\lim_{x\to 2}\left(\frac{2x^2+1}{x^2+6x-4}\right) = \frac{2(2)^2+1}{\cancel{2^2}+6(2)-\cancel{4}} = \frac{9}{12} = \boxed{\frac{3}{4}}$

b.) $\lim_{x\to -2}\sqrt{x^4+3x+6} = \sqrt{(-2)^4+\cancel{3(-2)}+\cancel{6}} = \sqrt{16} = \boxed{4}$

c.) $\lim_{x\to 3}\left(\frac{\cos(\pi x)(x-3)}{x^2-5x+6}\right) = \lim_{x\to 3}\left(\frac{\cos(\pi x)\cancel{(x-3)}}{(x-2)\cancel{(x-3)}}\right)$

$= \lim_{x\to 3}\left(\frac{\cos(\pi x)}{x-2}\right)$

$= \frac{\cos(3\pi)}{3-2} = \boxed{-1}$

4. (this was also on the practice test)

d.) $$\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right) = \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)\right)$$

$$= \lim_{h\to 0}\left(\frac{(\sqrt{x+h})^2 + \cancel{\sqrt{x+h}\sqrt{x}} - \cancel{\sqrt{x}\sqrt{x+h}} - (\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}\right)$$

$$= \lim_{h\to 0}\left(\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\right)$$

$$= \lim_{h\to 0}\left(\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

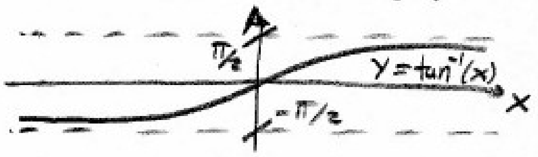
5. a.) $$\lim_{x\to\infty}\left(\frac{ax^2+bx+c}{x^2-2}\right) = \lim_{x\to\infty}\left(\frac{a+\frac{b}{x}+\frac{c}{x^2}}{1-\frac{2}{x^2}}\right)$$: dividing the numerator and denominator by x^2

$$= \boxed{a}$$ (the terms with $\frac{1}{x}$ or $\frac{1}{x^2}$ vanish as $x\to\infty$)

b.) $$\lim_{x\to\infty}\left(3\tan^{-1}(x)\right) = 3\lim_{x\to\infty}\left(\tan^{-1}(x)\right)$$ ← we discussed in lecture, see graph

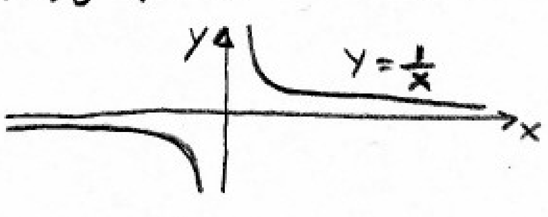
$$= 3\cdot\frac{\pi}{2}$$

$$= \boxed{\frac{3\pi}{2}}$$



c.) $$\lim_{x\to 0^-}\left(\frac{1}{x}\right) = \boxed{-\infty}$$

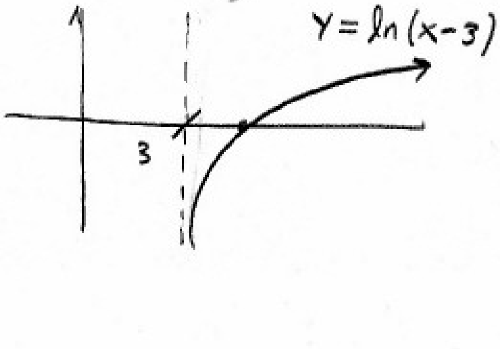
• we take values closer and closer to zero from the left, they are negative but small ⇒ $-\infty$.



← (or you could use the graph to see it.)

d.) $$\lim_{x\to 3^+}\left(\ln(x-3)\right) = \boxed{-\infty}$$

The graph is $\ln(x)$ shifted 3 units right.



6. $f'(x) \equiv \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$ OR $f'(x) \equiv \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$

7.
$$\frac{d}{dx}\left(\sin(x)\right) \equiv \lim_{h\to 0}\left(\frac{\sin(x+h)-\sin(x)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}\right)$$
$$= \lim_{h\to 0}\left(\sin(x)\left[\frac{\cos(h)-1}{h}\right]+\cos(x)\left[\frac{\sin(h)}{h}\right]\right)$$
$$= \sin(x)\underbrace{\lim_{h\to 0}\left[\frac{\cos(h)-1}{h}\right]}_{\to 0}+\cos(x)\underbrace{\lim_{h\to 0}\left[\frac{\sin(h)}{h}\right]}_{\to 1}$$
$$= \boxed{\cos(x)}$$

8. a.) $\frac{d}{d\theta}\left(\sin\theta+3\right) = \boxed{\cos\theta}$

b.) $\frac{d}{dx}\left(x(a\sqrt{x}+bx^2)\right) = \frac{d}{dx}\left(ax^{\frac{3}{2}}+bx^3\right) = \boxed{\frac{3a}{2}x^{\frac{1}{2}}+3bx^2}$

c.) $\frac{d}{dx}\left(ce^x-\frac{5}{\sqrt{x^3}}\right) = c\frac{d}{dx}(e^x)-5\frac{d}{dx}\left(x^{-\frac{3}{2}}\right) = \boxed{ce^x+\frac{15}{2}x^{-\frac{5}{2}}}$

d.) $\frac{d}{dx}\left(\frac{x^2+3}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{\frac{3}{2}}+3x^{-\frac{1}{2}}\right) = \boxed{\frac{3}{2}x^{\frac{1}{2}}-\frac{3}{2}x^{-\frac{3}{2}}}$

9. $f(x)=\sqrt{x} \Rightarrow f'(x)=\frac{1}{2\sqrt{x}}$. The tangent thru $(4,2)$ has slope $f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{4}$ thus $y=2+\frac{1}{4}(x-4)=\boxed{1+\frac{1}{4}x=y}$

Bonus: Given $f'(a)$ is well-defined show that f is continous at a.

Differentiable $\Rightarrow$ Continous

$$\lim_{x\to a}\left(f(x)-f(a)\right) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right)$$
$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\lim_{x\to a}(x-a)$$
$$= f'(a)\cdot 0$$
$$= 0 \implies \lim_{x\to a}(f(x)) = \lim_{x\to a}(f(a)) = f(a)$$