MA 141 - 003, TEST III, SUMMER II 2005

WRITE SOLUTIONS
ON SEPERATE SHEETS
KEEP THE TEST

1 (5pts.) Approximate $\sqrt{16.08}$ using an appropriate linearization.

2 (10pts.) Assume that a particle moves along the curve $Y = \sqrt{1 + x^3}$. If the X-coordinate is increasing at 1 cm/s at the point (2, 3) then how does Y change w.r.t. time at (1, 2)?

3 (15pts.) Calculate the limits, don't forget to indicate your reasoning (L

(a.) $\lim\limits_{\theta \to 0} \left(\frac{\tan(p\theta)}{\tan(q\theta)} \right)$

(b.) $\lim\limits_{x \to 0^+} \left(x e^{\frac{1}{x}} \right)$

(c.) $\lim\limits_{x \to 0^+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1} \right)$

(d.) $\lim\limits_{x \to \infty} \left(x^2 e^{-x} \right)$

(e.) $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$

4 (30pts.) Let $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x$. Find for $f(x)$,

a.) critical #'s

b.) intervals of inc./dec.

c.) local min/max values. (where they are and what they are.)

d.) intervals of concave up/down.

e.) inflection points

f.) graph using data from parts a.) → e.)

5 (15pts.) Find the point on $Y = 3x$ closest to (1, 0).
(Use calculus please, make sure to prove it is really the closest pt.)

$\boxed{6\ (10\text{pts.})}$ Let $f(x) = |x|$ then

a.) Calculate R_5 to approx. area under $Y = |X|$ on $[-2, 3]$.

b.) Calculate the area exactly.

c.) Graph R_5 and $\int_{-2}^{3} |x|\,dx$ by graphing $Y = |X|$ twice then on one graph draw the rectangles for R_5 and on the other shade in the area the integral yields.

$\boxed{7\ (15\text{pts.})}$ Calculate the integrals below. If it is a definite integral then evaluate and simplify. If it is indefinite please leave the most general antiderivative as your answer.

a.) $\int_0^1 (mx^3 + 2x - \pi)\,dx$

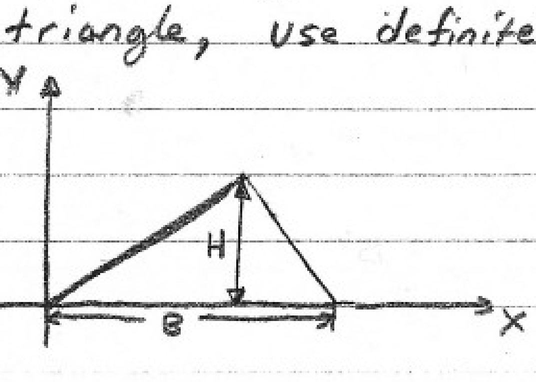
b.) $\int_{-8}^{-4} \left(\frac{1}{x}\right) dx$

c.) $\int (5^x + x^5 + \sqrt[3]{x})\,dx$

d.) $\int \left(\sec(x)\tan(x) + \frac{1}{1+x^2}\right) dx$

e.) $\int \left(e^x + \frac{13}{\sqrt{1-x^2}}\right) dx$

BONUS: PROVE THAT $A = \frac{1}{2}(\text{BASE})(\text{HEIGHT})$ gives the area of a triangle, use definite integration. Hint: ↙



SOLUTION TO TEST III

1.) Let $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$. We should center our approximation around 16. That is,

$$\sqrt{x} \cong \sqrt{16} + \frac{1}{2\sqrt{16}}(x-16)$$

$$\sqrt{x} \cong 4 + \tfrac{1}{8}(x-16)$$

Then $\sqrt{16.08} \cong 4 + \frac{1}{8}(16.08-16) = \boxed{4.01}$

2.) Given that $Y = \sqrt{1+x^3}$ and $\frac{dx}{dt} = 1\,\text{cm/s}$ at $(2,3)$ find $\frac{dY}{dt}$.

$$\frac{dY}{dt} = \frac{1}{2\sqrt{1+x^3}} \cdot 3x^2\frac{dx}{dt}$$ now plug in our data,

$$\frac{dY}{dt} = \frac{\cancel{3}(2)^2}{\cancel{2}\sqrt{1+8}} \cdot 1\tfrac{\text{cm}}{\text{s}} = \boxed{2\,\text{cm/s} = \frac{dY}{dt}}$$

the (1, 2) on test should have read (2, 3)

3.) a.) $$\lim_{\theta\to 0}\left(\frac{\tan(P\theta)}{\tan(q\theta)}\right) \underset{(\frac{0}{0})}{\overset{\mathcal{L}}{=}} \lim_{\theta\to 0}\left(\frac{\sec^2(P\theta)\cdot P}{\sec^2(q\theta)\cdot q}\right) = \frac{\sec^2(0)}{\sec^2(0)}\frac{P}{q} = \boxed{\frac{P}{q}}$$

b.) $$\lim_{x\to 0^+}\left(xe^{\frac{1}{x}}\right) = \lim_{x\to 0^+}\left(\frac{e^{\frac{1}{x}}}{\frac{1}{x}}\right) \underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x\to 0^+}\left(\frac{e^{\frac{1}{x}}\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}\right) = \lim_{x\to 0^+}\left(e^{\frac{1}{x}}\right) = \boxed{\infty}$$

c.) $$\lim_{x\to 0^+}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) = 0 - \frac{1}{0-1} = \boxed{1}$$

(this was a typo I meant for it to read $x \to 1$ instead, as such I graded this bonus. Since I promised all L'Hopitals.)

d.) $$\lim_{x\to\infty}\left(x^2e^{-x}\right) = \lim_{x\to\infty}\left(\frac{x^2}{e^x}\right)$$

$$\underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x\to\infty}\left(\frac{2x}{e^x}\right)$$

$$\underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x\to\infty}\left(\frac{2}{e^x}\right) = \boxed{0}$$

③ Continued

e.) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^{\underbrace{\lim_{n\to\infty}\left(n\ln\left(1+\frac{1}{n}\right)\right)}_{*}}$

$$* = \lim_{n\to\infty}\left(\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}\right) \underset{\left(\frac{0}{0}\right)}{\overset{\mathcal{L}}{=}} \lim_{n\to\infty}\left(\frac{\left(\frac{1}{1+\frac{1}{n}}\right)\cdot\frac{-1}{n^2}}{\frac{-1}{n^2}}\right) = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = 1.$$

Thus going back to top and plugging in $*$ gives $\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1}$

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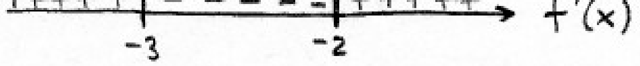
④ Let $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x$

a.) $f'(x) = x^2 + 5x + 6 = (x+3)(x+2) \Rightarrow$ $\boxed{\text{Critical \#'s } -3 \text{ and } -2}$

b.) 

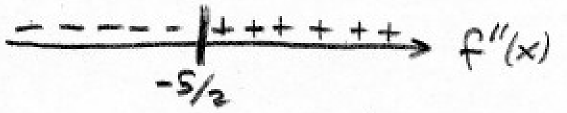
f is increasing on $(-\infty,-3)$ and $(-2,\infty)$
f is decreasing on $(-3,-2)$

c.) By Fermat's Thm we know max/min are only at critical #'s and By 1st derivative test and the sign chart for $f'(x)$ we can conclude there is a $\boxed{\text{max at } x=-3}$ and a $\boxed{\text{min. at } x=-2}$.

local max at -3 is $f(-3) = \frac{1}{3}(-27) + \frac{5}{2}(9) - 18 = \boxed{\frac{-9}{2}}$

local min at -2 is $f(-2) = \frac{1}{3}(-8) + \frac{5}{2}(4) - 12 = \boxed{-14/3}$

d.) $f''(x) = 2x + 5$ is only zero when $x = -5/2$

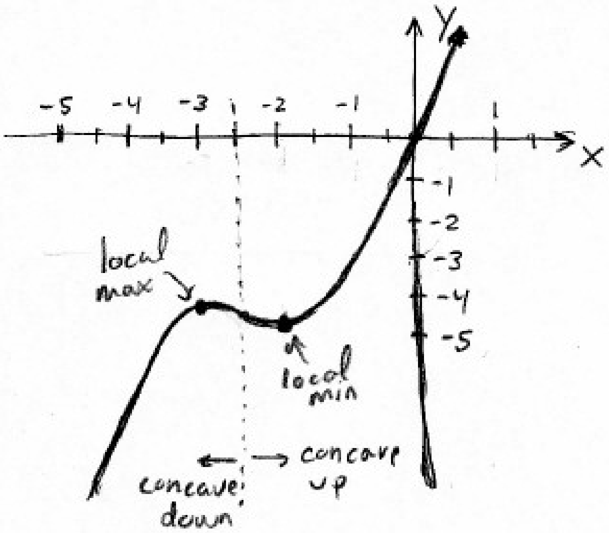


$\boxed{\begin{array}{l} f \text{ is concave down on } (-\infty,-5/2) \\ f \text{ is concave up on } (-5/2,\infty) \end{array}}$

e.) Inflection point is at $x = -5/2$

f.) on next page ↷

④ Now graph $Y = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x$,



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⑤ The distance s from (x, y) on $y = 3x$ and the point $(1, 0)$ is,

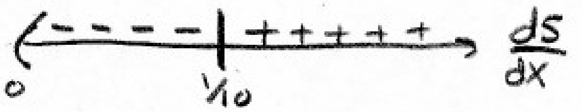
$$s(x) = \sqrt{(x-1)^2 + (3x-0)^2} = \sqrt{10x^2 - 2x + 1}$$

Now minimize the distance,

$$\frac{ds}{dx} = \frac{20x - 2}{2\sqrt{10x^2 - 2x + 1}}$$

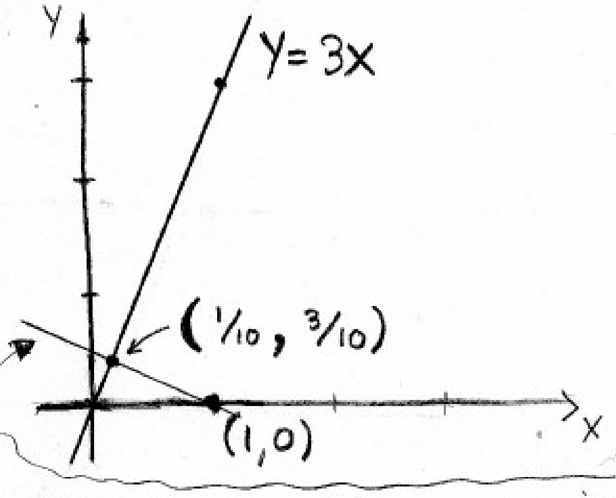
Only critical # is the sol^n to $20x - 2 = 0$, namely $\boxed{x = \frac{1}{10}}$

We need to show this is a minimum, use 1st Der. test



$\Rightarrow$ <u>$s(1/10)$ is a minimum</u>

The closest pt. is thus $(1/10, 3(\frac{1}{10})) = \boxed{(1/10, 3/10)}$. I didn't ask for it but lets graph the result and see if its reasonable,



Slope of this line is $\frac{\Delta Y}{\Delta X} = \frac{3/10 - 0}{1/10 - 1} = \frac{3/10}{-9/10} = \frac{-1}{3}$ indeed the closest point lies on the perpendicular bisector.

⑥ a.) $R_5 = \Delta x\left(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\right)$ where $\Delta x = \frac{5}{5} = 1$

$x_0 = -2$
$x_1 = -1$
$x_2 = 0$
$x_3 = 1$
$x_4 = 2$
$x_5 = 3$

$= 1\left(|-1| + |0| + |1| + |2| + |3|\right)$

$= \boxed{7 = R_5}$

b.) $$\int_{-2}^{3} |x|\,dx = \int_{-2}^{0} |x|\,dx + \int_{0}^{3} |x|\,dx$$

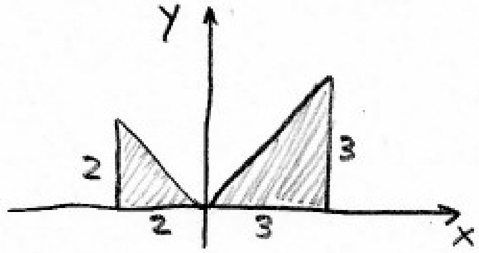
$$= \int_{-2}^{0} -x\,dx + \int_{0}^{3} x\,dx$$

$$= -\frac{1}{2}x^2\Big|_{-2}^{0} + \frac{1}{2}x^2\Big|_{0}^{3}$$

$$= \frac{1}{2}(-2)^2 + \frac{1}{2}(3)^2$$

$$= 2 + \frac{9}{2} = \boxed{\frac{13}{2} = 6.5}$$

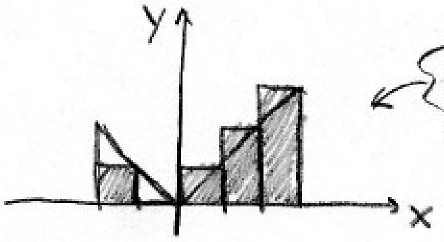
c.)



Another way to calc. the area in b. is,

$A = \frac{1}{2}(2)(2) + \frac{1}{2}(3)(3) = 2 + \frac{9}{2} = \boxed{6.5}$

Just added areas of the triangles.



the sum of the areas of these boxes gives R_5. Can you see why this was an overestimate of 0.5 from the true area?

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⑦ a.) $$\int_0^1 (mx^2 + 2x - \pi)\,dx = \left(\frac{m}{3}x^3 + x^2 - \pi x\right|_0^1$$

$$= \left(\frac{m}{3} + 1 - \pi\right) - \left(\frac{m}{3}(0) + 0 - \pi(0)\right)$$

$$= \boxed{\frac{m}{3} + 1 - \pi}$$

b.) $$\int_{-8}^{-4} \left(\frac{1}{x}\right)dx = \left(\ln|x|\right|_{-8}^{-4}$$

$$= \ln|-4| - \ln|-8|$$

$$= \boxed{\ln(4) - \ln(8)} = \ln(4/8) = \ln(1/2) = \ln(1) - \ln(2) = \boxed{-\ln(2)}$$

⑦ Continued

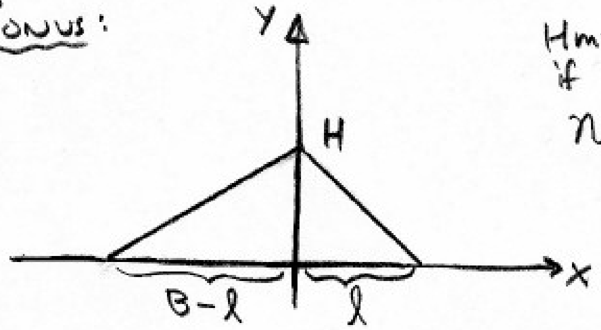
c.) $\int(5^x + x^5 + \sqrt[3]{x})\,dx = \boxed{\frac{1}{\ln(5)}5^x + \frac{1}{6}x^6 + \frac{3}{4}x^{4/3} + C}$

d.) $\int\left(\sec(x)\tan(x) + \frac{1}{1+x^2}\right)dx = \boxed{\sec(x) + \tan^{-1}(x) + C}$

e.) $\int\left(e^x + \frac{13}{\sqrt{1-x^2}}\right)dx = \boxed{e^x + 13\sin^{-1}(x) + C}$

Each of the above is correct because if you differentiate the answer you'll get back the integrand.

Bonus:



Hmm, it'd be easier to set-up the lines if I draw the triangle this way. Need to introduce ℓ.

Left line segment : $y = H + \left(\frac{H}{B-\ell}\right)x$ (the slope was clearly $\frac{H}{B-\ell}$)

Right line segment : $y = H - \left(\frac{H}{\ell}\right)x$ (the slope was $\frac{-H}{\ell}$, see it?)

Now integrate to find area, have to break into two-pieces,

$$A = \int_{-(B-\ell)}^{0}\left(H + \left(\frac{H}{B-\ell}\right)x\right)dx + \int_{0}^{\ell}\left(H - \frac{H}{\ell}x\right)dx$$

$$= \left(Hx + \frac{1}{2}\left(\frac{H}{B-\ell}\right)x^2\right|_{-(B-\ell)}^{0} + \left(Hx - \frac{1}{2}\frac{H}{\ell}x^2\right|_{0}^{\ell}$$

$$= -H(-(B-\ell)) - \frac{1}{2}\left(\frac{H}{B-\ell}\right)(B-\ell)^2 + H\ell - \frac{1}{2}\frac{H}{\ell}\ell^2$$

$$= HB - \cancel{H\ell} - \frac{1}{2}H(B-\ell) + \cancel{H\ell} - \frac{1}{2}H\ell$$

$$= HB - \frac{1}{2}HB + \cancel{\frac{1}{2}H\ell} - \cancel{\frac{1}{2}H\ell}$$

$$= \boxed{\frac{1}{2}HB = A}$$

notice that the ℓ we introduced cancels out in the end.

- I'm not advocating that this is the best way to prove this, the method from high-school geometry is alot easier. My point here is that we can derive such things with calculus.