PRACTICE TEST III : (APPLICATIONS OF DIFF. AND INTEGRATION)

$\boxed{1}\,\boxed{5\text{pts.}}$ Calculate $\sqrt{9.03}$ using an appropriate linearization.

$\boxed{2}\,\boxed{10\text{pts.}}$ Given that the volume V and radius r of a spherical balloon are related by $V = \frac{4}{3}\pi r^3$, find how fast the radius is changing when the volume is increasing at $10\,cm^3$ for $r = 2\,cm$.

$\boxed{3}\,\boxed{15\text{pts}}$ Use the notation we discussed in lecture as you calc. the limits,

a.) $\lim\limits_{x\to 0}\left(\frac{\sin(x)}{x}\right)$

b.) $\lim\limits_{x\to \infty}\left(xe^{-x}\right)$

c.) $\lim\limits_{x\to 1}\left(\frac{x^a - 1}{x^b - 1}\right)$

d.) $\lim\limits_{x\to 1}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right)$

e.) $\lim\limits_{x\to 0}\left(x^x\right)$

$\boxed{4}\,\boxed{30\text{pts.}}$ Credit will be given according to the completeness of answers.

Let $f(x) = x^3 + 2x^2 - 5x + 3$. Find then, for this fnct,

a.) critical #'s

b.) intervals of inc./dec.

c.) local min/max values.

d.) intervals of concave up/down

e.) inflection pts

f.) graph using (a) $\to$ (e).

In each part explain why what you found is correct and why there are no other answers (if appropriate to the part.)

$\boxed{5}\,\boxed{15\text{pts}}$ Find the point on $y = mx$ that is closest to $(1,0)$.

$\boxed{6}\,\boxed{10\text{pts.}}$ Let $f(x) = |x+2|$ then calculate,

a.) L_4 for $[-2,2]$

b.) the exact area using the FTC. (or a geometric argument)

$\boxed{7}\,\boxed{15\text{pts}}$ see next page

7 | 15 pts. Calculate the integrals below. If the integral is definite then evaluate and simplify. If the integral is indefinite leave the most general antiderivative as the answer.

a.) $\int_0^{\pi} (\sin(x) + e^x)\,dx$

b.) $\int_1^4 \left(\sqrt{x} + \sqrt[3]{x} + \frac{1}{\sqrt{x}}\right) dx$

c.) $\int \frac{3}{1+u^2}\,du$

d.) $\int (10^x + \csc^2(x))\,dx$

e.) $\int \left(\frac{9}{x} + \frac{6}{\sqrt{1-x^2}}\right) dx$

① What is $\sqrt{9.03}$ approx.?

$f(x)=\sqrt{x}$ + I need to find $L^{9}_{\sqrt{x}}(x)=f(9)+f'(9)(x-9)$

$$f'(x)=\frac{1}{2\sqrt{x}} \Rightarrow$$

$$\sqrt{x} \cong \sqrt{9}+\frac{1}{2\sqrt{9}}(x-9)=3+\frac{1}{6}(x-9) \cong \sqrt{x} \text{ (near } x=9)$$

$$\sqrt{9.03} \cong 3+\frac{1}{6}(9.03-9)$$
$$=3+\frac{1}{6}(.03)$$
$$\boxed{=3.005}$$

② Related Rates Problem:

$-V=\frac{4}{3}\pi r^3$

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ ⇒ I know when $r=2cm$ then $\frac{dV}{dt}=10cm^3/s$, need to find $\frac{dr}{dt}$

$$\frac{dr}{dt}=\frac{\frac{dV}{dt}}{4\pi r^2}=\boxed{\frac{10cm^3/s}{4\pi(2cm)^2}}$$

③ a) $\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) \underset{\left(\frac{0}{0}\right)}{\overset{\ell}{=}} \lim_{x\to 0}\left(\frac{\cos(x)}{1}\right)=\cos(0)=\boxed{1}$

b) $\lim_{x\to\infty}(xe^{-x})=\lim_{x\to\infty}\left(\frac{x}{e^x}\right)\overset{\ell}{=}\lim_{x\to\infty}\left(\frac{1}{e^x}\right)=\boxed{0}$

↳ 1 divided by anything large goes to zero.

③ c) $\lim_{x \to 1} \left(\frac{x^a - 1}{x^b - 1} \right) \overset{L}{=} \lim_{x \to 1} \left(\frac{a x^{a-1}}{b x^{b-1}} \right) = \frac{a}{b} \left(\frac{1^{a-1}}{1^{b-1}} \right) = \boxed{\frac{a}{b}}$

one to a constant $\left(\frac{0}{0}\right)$

d) $\lim_{x \to 1} \left(\frac{1}{\ln(x)} - \frac{1}{x-1} \right) = \lim_{x \to 1} \left(\frac{x - 1 - \ln(x)}{(x-1)\ln(x)} \right) =$

$\infty - \infty \therefore$
undefined

$\overset{L}{=} \lim_{x \to 1} \left(\frac{1 - \frac{1}{x}}{\ln(x) + (x-1)\frac{1}{x}} \right)$
$\left(\frac{0}{0}\right)$

↳ used product rule on bottom

$= \lim_{x \to 1} \left(\frac{1 - \frac{1}{x}}{\ln(x) + 1 - \frac{1}{x}} \right)$

$\overset{L}{=} \lim_{x \to 1} \left(\frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} \right) = \boxed{\frac{1}{2}}$
$\left(\frac{0}{0}\right)$

substitute

e) $\lim_{x \to 0^+} \left(x^x \right) = e^{\underbrace{\lim_{x \to 0^+} \left(\ln (x^x) \right)}_{*}}$

$* = \lim_{x \to 0^+} \left(x \left(\ln (x^x) \right) \right)$

$= \lim_{x \to 0^+} \left(\frac{\ln(x)}{\frac{1}{x}} \right)$

$\overset{L}{=} \lim_{x \to 0^+} \left(\frac{\frac{1}{x}}{-\frac{1}{x^2}} \right)$
$\left(\frac{\infty}{\infty}\right)$

$= \lim_{x \to 0^+} (-x) = 0$

$= e^0 = \boxed{1}$

Indeterminate power problem

④ $f(x) = x^3 + 2x^2 - 5x + 3$

a) critical #'s:

$f'(x) = 3x^2 + 4x - 5$

$f''(x) = 6x + 4$

will be prettier on test

are solutions to $f'(x) = 0$:

$$x = \frac{-4 \pm \sqrt{16 + 4(3)(5)}}{2(3)} \Rightarrow x = \frac{-4 \pm \sqrt{76}}{6} \text{ or } \boxed{\frac{-2}{3} \pm \frac{\sqrt{76}}{6}}$$

$\approx \boxed{.786 \text{ or } -2.12}$

b) $+++++ \mid - - - - - - - - \mid ++++++ \rightarrow f'(x)$

$-2.12 \qquad .786$

f is inc. on $(-\infty, -2.12) + (.786, \infty)$

f is dec. on $(-2.12, .786)$

c) can only look for local max/min @ critical points

→ max @ -2.12 by 1st deriv. test ⇒ $f(-2.12)$

→ min @ $.786$ by 1st derive. test ⇒ $f(.786)$

d) use 2nd deriv.: $- - - - - - - - \mid ++++++++ \rightarrow f''(x)$

$-2/3$

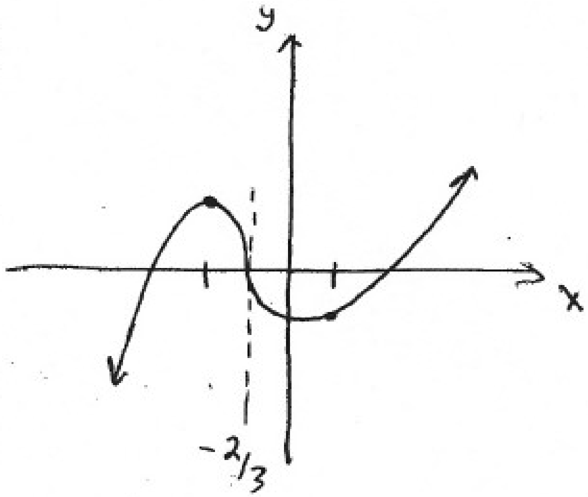
f is concave dn. on $(-\infty, -2/3)$

f is concave up. on $(-2/3, \infty)$

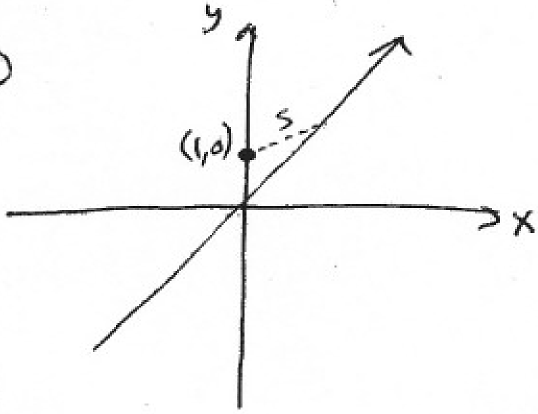
e) where is the Δ concavity?

↳ $x = -2/3$

f)



(5)



$s = \sqrt{(x-1)^2 + (y-0)^2}$

$s(x) = \sqrt{(x-1)^2 + (y)^2}$

$\boxed{s(x) = \sqrt{(x-1)^2 + (mx)^2}}$

↳ minimize this distance

we stopped here, let me continue, lets minimize $S(x)$.

$$\frac{dS}{dx} = \frac{1}{2\sqrt{(x-1)^2+(mx)^2}}\frac{d}{dx}\left((x-1)^2+(mx)^2\right) = \frac{x-1+m^2x}{\sqrt{(x-1)^2+(mx)^2}} = S'(x)$$

Notice that $(x-1)^2+(mx)^2 = x^2-2x+1+m^2x^2 = (1+m^2)x^2-2x+1$

- When is this quadratic zero? When $x = \frac{2\pm\sqrt{4-4(1+m^2)}}{2(1+m^2)}$ the quadratic is zero, simplifying $x = \frac{1\pm\sqrt{1-1-m^2}}{1+m^2} = \frac{1\pm\sqrt{-m^2}}{1+m^2} = \frac{1\pm i\sqrt{m}}{1+m^2}$.

- As we could have argued geometrically from the beginning the distance S from $y=mx$ to $(1,0)$ cannot be zero. Thus all critical #'s must fall from $S'(x)=0$,

$$x-1+m^2x = 0$$
$$x(1+m^2) = 1$$
$$\boxed{x = \frac{1}{1+m^2}}$$ only critical #.

$$\underset{\frac{1}{1+m^2}}{- - - - - - - |+++++++} \rightarrow S'(x)$$

$\therefore$ By 1st derivative test $S\left(\frac{1}{1+m^2}\right)$ is minimum distance.

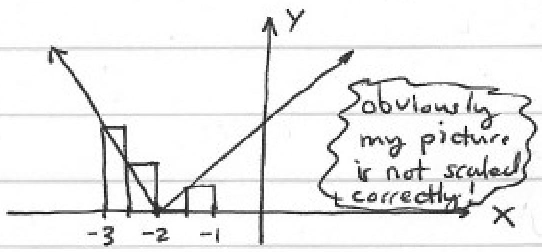
⑤ Continued, we just proved there is a minimum of $S(x)$ at $x = \frac{1}{1+m^2}$. Which point is closest? Lets finish the problem, $y = mx = m\left(\frac{1}{1+m^2}\right)$ thus

$\boxed{\left(\frac{1}{1+m^2}, \frac{m}{1+m^2}\right) \text{ is the closest point to } (1,0) \text{ on the line } y = mx.}$

⑥ Let $f(x) = |x+2|$ then calculate L_4 for $[-3,-1]$
(I changed it from $[-2,2]$ to make it interesting, oops.)

a.)
$$L_4 = \Delta x\left(f(x_0) + f(x_1) + f(x_2) + f(x_3)\right)$$
$$= \frac{-1-(-3)}{4}\left(|-3+2| + |-2.5+2| + |-2+2| + |-1.5+2|\right)$$
$$= \frac{1}{2}(1 + 0.5 + 0 + 0.5)$$
$$= \frac{1}{2}(2) = \boxed{1}$$

Graphically what we calculated looks like
Its absolute value shifted left 2 units.



b.)
$$|x+2| = \begin{cases} x+2 & x \geq -2 \\ -x-2 & x \leq -2 \end{cases}$$

$$\int_{-3}^{-1} |x+2|\,dx = \int_{-3}^{-2} |x+2|\,dx + \int_{-2}^{-1} |x+2|\,dx$$

$$= \int_{-3}^{-2} (-x-2)\,dx + \int_{-2}^{-1} (x+2)\,dx$$

APPLIED THE F.T.C.

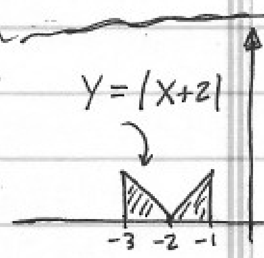
$$= \left(-\frac{x^2}{2} - 2x\right)\Big|_{-3}^{-2} + \left(\frac{x^2}{2} + 2x\right)\Big|_{-2}^{-1}$$

$$= \left(-\frac{(-2)^2}{2} - 2(-2)\right) - \left(-\frac{(-3)^2}{2} - 2(-3)\right) + \left(\frac{(-1)^2}{2} + 2(-1)\right) - \left(\frac{(-2)^2}{2} + 2(-2)\right)$$

$$= (-2+4) - \left(-\frac{9}{2} + 6\right) + \left(\frac{1}{2} - 2\right) - (2-4)$$

$$= (2) - \frac{3}{2} - \frac{3}{2} + 2$$

$$= \frac{8}{2} - \frac{6}{2} = \boxed{1}$$ apparently L_4 was a very good approx.



2 triangles with Area A,
$A = \frac{1}{2}bh$
$= \frac{1}{2}(1)(1)$
$= \frac{1}{2}$

Total Area $2A = 1$
(Geometric Shortcut)

7 a.) $\int_0^{\pi} (\sin(x) + e^x)\,dx = \left[-\cos(x) + e^x\right]_0^{\pi}$

$= (-\cos(\pi) + e^{\pi}) - (-\cos(0) + e^0)$

$= (1 + e^{\pi}) - (-1 + 1)$

$= \boxed{1 + e^{\pi}}$

b.) $\int_1^4 \left(\sqrt{x} + \sqrt[3]{x} + \frac{1}{\sqrt{x}}\right)dx = \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{3}{4}x^{\frac{4}{3}} + 2\sqrt{x}\right]_1^4$

$= \left(\frac{2}{3}(4)^{3/2} + \frac{3}{4}(4)^{\frac{4}{3}} + 2\sqrt{4}\right) - \left(\frac{2}{3} + \frac{3}{4} + 2\right)$

$= \left(\frac{2}{3}(8) + \frac{3}{4}(4)^{4/3} + 4\right) - \left(41/12\right)$

$= \boxed{3\sqrt[3]{4} + 71/12 = 10.68}$

c.) $\int \frac{3}{1+u^2}\,du = 3\int \frac{du}{1+u^2}$

$= \boxed{3\tan^{-1}(u) + C}$

d.) $\int (10^x + \csc^2(x))\,dx = \int 10^x\,dx + \int \csc^2(x)\,dx$

$= \boxed{\frac{1}{\ln(10)}10^x - \cot(x) + C}$

e.) $\int \left(\frac{a}{x} + \frac{b}{\sqrt{1-x^2}}\right)dx = a\int \frac{dx}{x} + b\int \frac{dx}{\sqrt{1-x^2}}$

$= \boxed{a\ln|x| + b\sin^{-1}(x) + C}$