INTEGRATION BY PARTS (IBP)

Basically I.B.P is the analogue to the product rule for $\int$'s.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \Rightarrow u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Now integrate,

$$\int u\frac{dV}{dx}dx = \int \frac{d}{dx}(uv)\,dx - \int V\frac{du}{dx}dx$$

$$\therefore \boxed{\int u\,dV = uV - \int v\,du}$$

E1 $\int xe^x dx = \int u\,dV = xe^x - \int e^x dx$

$= \boxed{xe^x - e^x + C}$

$u = x$	$dV = e^x dx$
$du = dx$	$V = e^x$

E2 $\int x\cos(x)dx = x\cos(x) - \int \sin(x)\,dx$

$= \boxed{x\cos(x) + \cos(x) + C}$

$u = x$	$dV = \cos(x)dx$
$du = dx$	$V = \sin(x)$

E3 $\int \ln(x)\,dx = x\ln(x) - \int x\frac{dx}{x}$

$= x\ln(x) - x + C$

$= \boxed{x(\ln(x) - 1) + C}$

$u = \ln(x)$	$dV = dx$
$du = \frac{dx}{x}$	$V = x$

E4 $\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}dx$

$= x\tan^{-1}(x) - \frac{1}{2}\int\frac{dw}{w}$

$= x\tan^{-1}(x) - \frac{1}{2}\ln|w| + c$

$= \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C}$

$u = \tan^{-1}(x)$	$dV = dx$
$du = \frac{1}{1+x^2}dx$	$V = x$

$w = 1+x^2$
$dw = 2x\,dx$

notice $1+x^2 > 0$ so we can drop the absolute value bars.

Integration by Parts, how to choose u

If we examine $\boxed{E1} \to \boxed{E4}$ on 112 we can note there is some pattern in choosing u:

Logarithmic function
Inverse function
Algebraic function
Trigonometric function
Exponential function

- listed in order of choosing u.

For many examples (if it appears to be IBP) we can choose u by simply going down this list and making u whatever we see first. Lets see how this worked

$\boxed{E1}$ $\int xe^x dx$ has x and e^x which are algebraic and exponential functions respectively.

L ⇒ no log or inverse
I
A ← 1st our example has one of ⇒ $u = x$
T
E

$\boxed{E3}$ $\int \ln(x)\,dx$ has a logarithm ⇒ L I A T E ⇒ chose $u = \ln(x)$

$\boxed{E4}$ $\int \tan^{-1}(x)dx$ has inverse function (& no log. fnct.) ⇒ L I A T E ⇒ $u = \tan^{-1}(x)$.

- I don't expect you to explain how you choose u, I mention, this as a useful heuristic rule, nothing more. B.T.W. this works for some u-subst. problems.

E5

$$\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx$$

$u = x^3$	$dv = e^x dx$
$du = 3x^2dx$	$v = e^x$

$$= x^3e^x - 3\left[x^2e^x - 2\int xe^x\,dx\right]$$

$u = x^2$	$dv = e^x dx$
$du = 2xdx$	$v = e^x$

$$= x^3e^x - 3\left[x^2e^x - 2\left(xe^x - \int e^x dx\right)\right]$$

$u = x$	$dv = e^x dx$
$du = dx$	$v = e^x$

$$= x^3e^x - 3\left[x^2e^x - 2xe^x + 2e^x\right] + C$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

$$= \boxed{e^x(x^3 - 3x^2 + 6x - 6) + C}$$

E6

$$\int e^x\cos(x)\,dx = e^x\cos(x) - \int \sin(x)\,e^x dx$$

$u = e^x$	$dv = \cos(x)\,dx$
$du = e^x dx$	$v = \sin(x)$

$$= e^x\cos(x) - \left[\sin(x)e^x + \int e^x\cos(x)\,dx\right]$$

$u = e^x$	$dv = \sin(x)$
$du = e^x$	$v = -\cos(x)$

$$= e^x(\cos(x) - \sin(x)) - \int e^x\cos(x)\,dx$$

∴ we've come full-circle back to where we began.

$$\Rightarrow 2\int e^x\cos(x)\,dx = e^x(\cos(x) - \sin(x))$$

$$\Rightarrow \boxed{\int e^x\cos(x)\,dx = \tfrac{1}{2}e^x(\cos(x) - \sin(x)) + C}$$

E7

$$\int \sin^{-1}(x)\,dx = \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$u = \sin^{-1}(x)$	$dv = dx$
$du = \frac{dx}{\sqrt{1-x^2}}$	$v = x$

$$= \sin^{-1}(x) + \frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$u = 1 - x^2$

$du = -2x\,dx \therefore \frac{du}{2} = -x\,dx$

$$= \sin^{-1}(x) + \sqrt{u} + C$$

$$= \boxed{\sin^{-1}(x) + \sqrt{1-x^2} + C}$$

E8

$$I = \int \cos(\ln(x))\,dx = x\cos(\ln(x)) + \int \sin(\ln(x))\,dx$$

$u = \cos(\ln(x))$	$dV = dx$
$du = \frac{-\sin(\ln(x))\,dx}{x}$	$V = x$

$$= x\cos(\ln(x)) + \left[x\sin(\ln(x)) - \int \cos(\ln(x))\,dx\right]$$

$u = \sin(\ln(x))$	$dV = dx$
$du = \frac{\cos(\ln(x))\,dx}{x}$	$V = x$

$$= x\big(\cos(\ln(x)) + \sin(\ln(x))\big) - I$$

$$\therefore \boxed{I = \tfrac{1}{2}\,x\big(\cos(\ln(x)) + \sin(\ln(x))\big) + C}$$

E9

$$\int \cos^n(x)\,dx = \int \cos^{n-1}(x)\cos(x)\,dx$$

$u = \cos^{n-1}(x)$
$du = (n-1)\cos^{n-2}(x)(-\sin(x))$
$dV = \cos(x)\,dx$
$V = \sin(x)$

$$= \cos^{n-1}(x)\sin(x) - \int \sin(x)(n-1)\cos^{n-2}(x)(-\sin(x))$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \sin^2(x)\cos^{n-2}(x)\,dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int (1-\cos^2(x))\cos^{n-2}(x)\,dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\left(\int \cos^{n-2}(x)\,dx + \int \cos^n(x)\,dx\right)$$

$$\therefore \boxed{\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx}$$

Solving for the integral algebraically.

Let $n=2$ then we find:

$$\int \cos^2(x)\,dx = \tfrac{1}{2}\cos(x)\sin(x) + \tfrac{1}{2}\int dx = \boxed{\tfrac{1}{2}\big(\cos(x)\sin(x) + x\big) + C}$$

Let $n=3$ then:

$$\int \cos^3(x)\,dx = \frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\int \cos(x)\,dx$$

$$= \frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\sin(x) + C$$

$$= \boxed{\tfrac{1}{3}\sin(x)\big(\cos^2(x) + 2\big) + C}$$

this is problem 34 a & b

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Remark: this is the other popular method to compute integrals of powers of cosine, there is also a similar "recurrence" relation for $\sin\theta$.