Springs with Damping and RLC Circuits

Consider a mass moving thru a viscous media, a good model to the friction force is $-c\frac{dx}{dt} = F_{friction}$, as before $-kx = F_{spring}$ thus in the absense of other forces Newton's 2^{nd} Law reads:

$$ma = -cv - kx$$

Which is using $\dot{x} = \frac{dx}{dt} = v$ & $\ddot{x} = \frac{d^2x}{dt^2} = a$ simply an 2^{nd} order Linear ODE Eq.^{n} with constant coefficients,

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0 \quad Eq^n\ (1)}$$

We then have the characteristic Eq.^{n} $m\lambda^2 + c\lambda + k = 0$ with sol^n's

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Which depending on the values of m, c, k gives sol^n's to Eq.^{n} (1)

Physical Description	Form of Sol^n	Which Case From Before
Over Damping	$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$	I.) $c^2 - 4mk > 0$
Critical Damping	$x = c_1 e^{\lambda t} + c_2 x e^{\lambda t}$	II.) $c^2 - 4mk = 0$
Under Damping	$x = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$	III.) $c^2 - 4mk < 0$ $\alpha = \frac{-c}{2m}$ $\beta = \frac{\sqrt{c^2 - 4mk}}{2m}$

E1 Suppose we have a 5 kg mass secured to a spring with $k = 10\frac{N}{m}$ if we place the spring & mass in a liquid with $c = 2\,kg/s$, describe the resulting motion.

$$5\ddot{x} + 2\dot{x} + 10x = 0 \Rightarrow 5\lambda^2 + 2\lambda + 10 = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 200}}{10} = \frac{-1}{5} \pm i\frac{\sqrt{196}}{10} = \frac{-1}{5} \pm \frac{7}{5}i$$

Thus $x(t) = e^{-0.2t}\left(c_1 \cos(1.4t) + c_2 \sin(1.4t)\right)$. Further suppose that the mass started at the equilibrium position with velocity $\dot{x}(0) = 1.4$ find the eq^n of motion. First find the velocity at time t,

$$\dot{x} = \frac{dx}{dt} = -0.2e^{-0.2t}\left(c_1\cos(1.4t) + c_2\sin(1.4t)\right) + e^{-0.2t}\left(-1.4c_1\sin(1.4t) + 1.4c_2\cos(1.4t)\right)$$

Now use the intial conditions;

$$x(0) = e^0\left(c_1\cos(0) + c_2\sin(0)\right) = \boxed{c_1 = 0}$$

$$\dot{x}(0) = -0.2e^0(\underbrace{c_1}_{0}\cos(0) + c_2\underbrace{\sin(0)}_{0}) + e^0(-1.4c_1\underbrace{\sin(0)}_{0} + 1.4c_2\cos(0)) = 1.4c_2 = 1.4$$

$\therefore \boxed{c_2 = 1}$ $\qquad \therefore \boxed{x(t) = e^{-0.2t}\sin(1.4t)} \leftarrow$ Eq^n of Motion

FORCED OSCILLATIONS

Suppose that in addition to the damping force we have some motor pushing & pulling on the spring; $F(t)$ then we have Newton's 2^{nd} Law:

$$ma = -kx - c\dot{x} + F(t)$$

Which gives a non-homogeneous 2^{nd} order ODEq^{n}:

$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t) \qquad Eq^{n}\ (2)}$$

Now for certain functions $F(t)$ we have the technology to solve this.

$\boxed{E2}$ Let $m = 1kg$ and $k = 25N/m$ with $c = 0$ and apply a force $F(t) = F_0\cos(5t)$

$$\ddot{x} + 25x = F_0\cos(5t)$$

Find the resulting motion if $x(0) = 10$ and $\dot{x}(0) = 0$. Find complementary sol^{n}:

$$\lambda^2 + 25 = 0 \Rightarrow \lambda = \pm 5i \Rightarrow X_c(t) = C_1\cos(5t) + C_2\sin(5t)$$

Notice X_c overlaps the forcing term so guess: $X_p = t(A\cos(5t) + B\sin(5t))$

$$\dot{X}_p = A\cos(5t) + B\sin(5t) + t(-5A\sin(5t) + 5B\cos(5t))$$
$$= \cos(5t)[A + 5Bt] + \sin(5t)[B - 5At]$$

$$\ddot{X}_p = -5\sin(5t)[A + 5Bt] + 5B\cos(5t)$$
$$+5\cos(5t)[B - 5At] - 5A\sin(5t)$$
$$= \cos(5t)[10B - 25At] + \sin(5t)[-10A - 25Bt]$$

Substituting into $\ddot{X}_p + 25X_p = F_0\cos(5t)$ to determine A, B gives,

$$\cos(5t)[10B - 25At] + \sin(5t)[-10A + 25Bt] + 25t(A\cos(5t) + B\sin(5t)) = F_0\cos(5t)$$

$$\left.\begin{array}{ll} \cos(5t): & 10B = F_0 \\ \sin(5t): & -10A = 0 \end{array}\right\} \Rightarrow A = 0 \ \& \ B = F_0/10$$

$$\left.\begin{array}{ll} t\cos(5t): & -25A + 25A = 0 \\ t\sin(5t): & 25B - 25B = 0 \end{array}\right\} \text{no info.}$$

The general sol^{n} is thus $X(t) = C_1\cos(5t) + C_2\sin(5t) + \frac{F_0}{10}\cos(5t)\cdot t$

Then $\dot{X}(t) = -5C_1\sin(5t) + 5C_2\cos(5t) - \frac{F_0}{2}\sin(5t) + \frac{F_0}{10}\cos(5t)$

$$X(0) = C_1 + F_0/10 = 10 \Rightarrow \boxed{C_1 = 10 - F_0/10}$$
$$\dot{X}(0) = 5C_2 + F_0/10 = 0 \quad \boxed{C_2 = -F_0/50}$$

Therefore $\boxed{X(t) = \left(10 - \frac{F_0}{10}\right)\cos(5t) - \frac{F_0}{50}\sin(5t) + \frac{F_0}{10}t\cos(5t)}$

Remark: $X \longrightarrow \infty$ as $t \longrightarrow \infty$ (the spring will break! (resonance)

E3 Forced - Damped - oscillator (No resonance ü)

$$\ddot{x} + 5\dot{x} + 6x = \sin(t) \quad \text{with } x(0) = 0 \;\&\; \dot{x}(0) = 3$$

Startegy : ① find x_c ② find x_p ③ Assemble $x = x_c + x_p$ and apply intial conditions,

① $\lambda^2 + 5\lambda + 6 = 0 \Rightarrow (\lambda+3)(\lambda+2) = 0 \therefore \lambda = -2, \lambda = -3$

$$x_c = c_1 e^{-2t} + c_2 e^{-3t}$$

② Clearly no overlap so

$$x_p = A\sin t + B\cos t$$
$$\dot{x}_p = A\cos t - B\sin t$$
$$\ddot{x}_p = -A\sin t - B\cos t = -x_p$$

Subst. into ODE$_q^{n}$:

$$\ddot{x}_p + 5\dot{x}_p + 6x_p = -x_p + 5(A\cos t - B\sin t) + 6x_p$$
$$= 5(A\sin t + B\cos t) + 5(A\cos t - B\sin t)$$
$$= (5A - 5B)\sin t + (5B + 5A)\cos t = \sin t$$

Thus comparing coefficients,

$$5A - 5B = 1 \Rightarrow A = \tfrac{1}{5}(5B + 1) = B + 1/5$$
$$5B + 5A = 0 \Rightarrow A = -B$$

That is $B + 1/5 = -B \Rightarrow 2B = -1/5 \therefore \boxed{B = -1/10 \;\&\; A = 1/10}$

③ Gen. Sol^{n}: $x(t) = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{10}(\sin t - \cos t)$

$$\dot{x}(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t} + \frac{1}{10}(\cos t + \sin t)$$

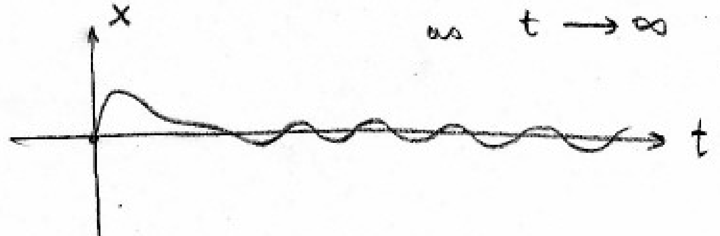
$$x(0) = c_1 + c_2 - \frac{1}{10} = 0 \Rightarrow c_1 = \frac{1}{10} - c_2$$

$$\dot{x}(0) = -2c_1 - 3c_2 + \frac{1}{10} = 3 \Rightarrow \underbrace{-2\left(\frac{1}{10} - c_2\right) - 3c_2 + \frac{1}{10} = 3}$$

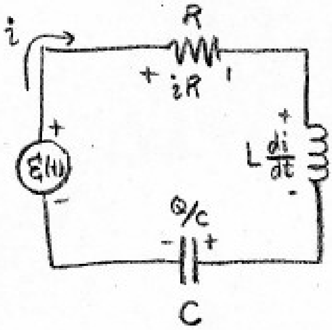
$$-c_2 = \frac{30}{10} - \frac{1}{10} + \frac{2}{10} = \frac{31}{10} \therefore c_2 = \frac{-31}{10}$$

$$c_1 = \frac{1}{10} + \frac{31}{10} = \frac{32}{10} = c_1$$

$$\boxed{x(t) = 3.2e^{-2t} - 3.1e^{-3t} + 0.1(\sin(t) - \cos(t))}$$



as $t \to \infty$ $x \to x_p$ ← the steady state Sol^{n}



Kirchoff's Law: energy is conserved.

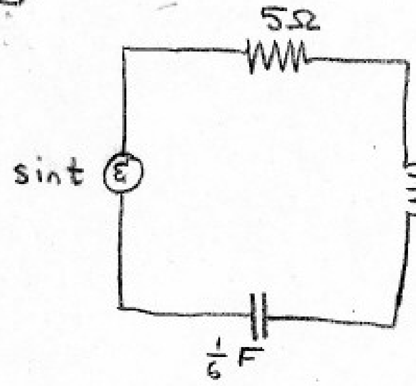
$$iR + L\frac{di}{dt} + \frac{Q}{C} = \mathcal{E}(t)$$

Where $i = \frac{dQ}{dt}$ the rate of charge passing thru the circuit.

So we are again faced with a non-homogeneous 2nd order ODE,

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}(t)}$$

$\boxed{E4}$ Find the current for the following circuit if $Q(0) = 0$, $i(0) = 3$,



$$\underbrace{\frac{d^2Q}{dt^2} + 5\frac{dQ}{dt} + 6Q = \sin(t)}_{\text{Same math as } \boxed{E3}\,!}$$

$$\boxed{Q(t) = 3.2e^{-2t} - 3.1e^{-3t} + 0.1(\sin t - \cos t)}$$

But remember we want $i(t) = \frac{dQ}{dt}$ so differentiate:

$$\boxed{i(t) = -6.4e^{-2t} + 9.3e^{-3t} + 0.1(\cos t + \sin t)}$$

The steady state sol$^{\underline{n}}$ is $i_p = 0.1(\cos t + \sin t)$ (as $t \to \infty$)

Comparison of RLC & Damped-Forced Oscillators

Springs		RLC Circuits	
x	displacement	Q	charge
$\dot{x}$	velocity	$i = \dot{Q}$	current
m	mass	L	inductance
c	damping const.	R	resistance
k	spring const.	$1/c$	recip. of capacitance
$F(t)$	Forcing Force	$\mathcal{E}(t)$	Source Voltage

Coulombs = C

$\frac{\text{Coulombs}}{\text{Second}}$ = Ampere = A

Henrys = H

Ohms = Ω

{Capacitance} = F = Farads.

Voltage = $\frac{\text{Pot. Energy}}{\text{unit charge}}$

$$V = Q/C \rightarrow V = \frac{C}{F}$$

$$V = \frac{J}{C}$$