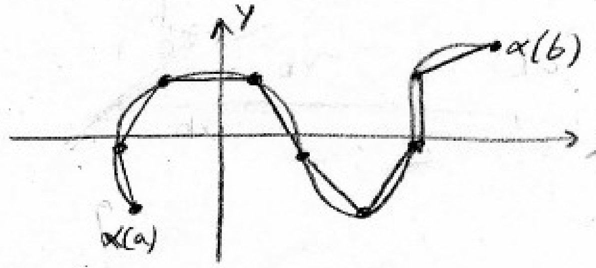
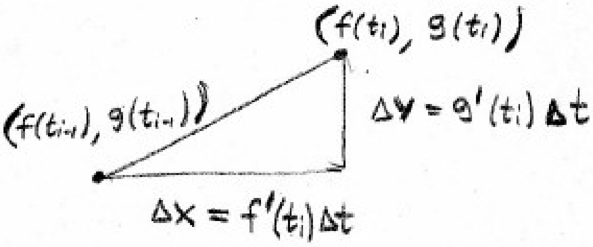
ARCLENGTH

We are given some parametrized curve $\alpha : I \to \mathbb{R}^2$ the xy-plane. That is $\alpha(t) = (f(t), g(t))$ $\forall t \in I \subset \mathbb{R}$.
Lets picture it : Let $I = [a, b]$



if we calculated the length of each line segments and added-em up that would give some estimation of the "arclength"

Divide $[a, b]$ into n - subintervals $[t_{i-1}, t_i]$ with $t_i = a + i\Delta t$ & $\Delta t = \frac{b-a}{n}$
then the length of a particular segment



$$\Delta S_i = \sqrt{(\Delta X_i)^2 + (\Delta Y_i)^2}$$
$$= \sqrt{(f'(t_i))^2 (\Delta t)^2 + (g'(t_i))^2 \Delta t^2}$$
$$= \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\, \Delta t$$

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta S_i$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\, \Delta t$$

$$\boxed{S = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\, dt}$$

- Derivation I like $\alpha(t) = (x(t), y(t))$ ← a parametrized curve

$$ds^2 = dx^2 + dy^2 = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) dt^2$$

$$\therefore \quad \boxed{S = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Remark when we parametrize by x then $\frac{dx}{dx} = 1$ and

$$\boxed{S = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

$\boxed{E1}$ Let $\alpha(t) = (r\cos t, r\sin t)$ $\quad 0 \le t \le 2\pi$

$x = r\cos t \qquad \frac{dx}{dt} = -r\sin t \quad \therefore \quad \left(\frac{dx}{dt}\right)^2 = r^2\sin^2 t$

$y = r\sin t \qquad \frac{dy}{dt} = r\cos t \quad \therefore \quad \left(\frac{dy}{dt}\right)^2 = r^2\cos^2 t$

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt = \int_0^{2\pi} r\, dt = \boxed{2\pi r}$$

Which is good since $x^2+y^2=r^2$, is a circle of radius r.

$\boxed{E2}$ Let $\alpha(x) = (x, \ln(\cos(x)))$ $\quad 0 \le x \le \pi/4$ $\quad$ $x = x$, $y = \ln(\cos(x))$

$y = \ln(\cos(x))$ for $0 \le x \le \pi/4$

$$S = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \left(\frac{-\sin(x)}{\cos(x)}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2(x)}\, dx \quad ; \quad \tan^2(x) + 1 = \sec^2(x)$$

$$= \int_0^{\pi/4} \sec(x)\, dx$$

$$= \int_1^{1+\sqrt{2}} \frac{du}{u}$$

$u = \sec(x) + \tan(x)$

$\frac{du}{u} = \sec(x)\,dx$

$u(0) = \sec(0) + \tan(0) = 1$

$u(\pi/4) = \sec(\pi/4) + \tan(\pi/4) = \sqrt{2} + 1$

$$= \left(\ln|u|\right)\Big|_1^{1+\sqrt{2}}$$

$$= \ln(1+\sqrt{2}) - \ln(1)$$

$$= \boxed{\ln(1+\sqrt{2})}$$

E3 Find circumference of an ellipse: $x = a\cos t$, $y = b\sin t$; $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \sin^2 t + \cos^2 t = 1$

$\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$ where we are given $0 \le t \le 2\pi$.

$$S = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} a\sqrt{\sin^2 t + \left(\frac{b\cos t}{a}\right)^2}\, dt$$

$$= \int_0^{2\pi} a\sqrt{1 - \cos^2 t + \left(\frac{b\cos t}{a}\right)^2}\, dt$$

$$= \int_0^{2\pi} a\sqrt{1 - (1-(b/a)^2)\cos^2 t}\, dt \quad : \text{ define } \beta = \sqrt{|1-(b/a)^2|}$$

$$= \int_0^{2\pi} a\sqrt{1 - \beta^2\cos^2 t}\, dt$$

This integral is not elementary. We'll need a specific a and b in order to proceed. Note that when $a = b$ then $\beta = 0$ so $S = 2\pi a$ happily. Consider $a = 1$, $b = \sqrt{2}$ then $\beta = 1$

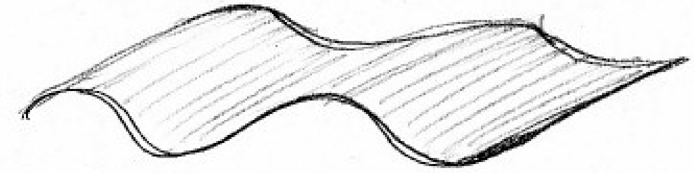
$$\int_0^{2\pi} \sqrt{1-\cos^2 t}\, dt =$$

$$= \int_0^{2\pi} |\sin t|\, dt = \int_0^{\pi} \sin t\, dt - \int_{\pi}^{2\pi} \sin t\, dt = 4$$

For most choices of a & b we would need to do this integral numerically (use a calculator, maple or the methods in §5.9)

E4 Produce panels 28" wide & 2" thick. From the picture the panels have 2 full-waves so $2\lambda = 28''$ and amplitude 2"

$y = \sin(\pi x/7)$ should have $4\pi = \frac{\pi}{7} \cdot 28$ ✓



$$\frac{dy}{dx} = \frac{\pi}{7}\cos\left(\frac{\pi x}{7}\right)$$

$$S = \int_0^{28} \sqrt{1 + \left(\frac{\pi}{7}\cos\left(\frac{\pi x}{7}\right)\right)^2}\, dx = \boxed{29.36''}$$

E5 Consider the following parametric curve where $0 \le t \le 3$, find arclength.

$x = e^t + e^{-t}$ $\quad \dot{x} = e^t - e^{-t}$ $\quad \dot{x}^2 + \dot{y}^2 = (e^t - e^{-t})^2 + 4$

$y = 5 - 2t$ $\quad \dot{y} = -2$

$$s = \int_0^3 \sqrt{e^{2t} - 2 + e^{-2t} + 4}\, dt = \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt = \int_0^3 (e^t + e^{-t})\, dt$$

$$= (e^t - e^{-t})\Big|_0^3$$

$$= \boxed{e^3 - 1/e^3}$$

E6 $y = \frac{x^3}{6} + \frac{1}{2x}$ with $1/2 \le x \le 1$ find arclength.

$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$ $\therefore$ $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)$

$$s = \int_{1/2}^1 \sqrt{1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)}\, dx = \int_{1/2}^1 \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\, dx = \int_{1/2}^1 \sqrt{\frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2}\, dx$$

$$= \int_{1/2}^1 \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_{1/2}^1 = \frac{31}{48} = \boxed{0.6458}$$