Average Value of a Function:

If we take the area under $y=f(x)$ on $a \leq x \leq b$ and divide by the width of the interval which is $b-a$ this gives the average,

$$f_{ave} \equiv \frac{1}{b-a}\int_a^b f(x)\,dx$$

this is the natural generalization to discrete averages (see pg. 473)

E1 Let $f(x) = \sin^3(x)$ find f_{ave} on $[0, \pi]$

$$f_{ave} = \frac{1}{\pi}\int_0^{\pi} \sin^3(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(1-\cos^2(x)\right)\sin(x)\,dx$$

$$= \frac{1}{\pi}\int_1^{-1} (1-u^2)(-du)$$

$u = \cos(x)$
$du = -\sin(x)\,dx$
$u(0) = \cos(0) = 1$
$u(\pi) = \cos(\pi) = -1$

$$= \frac{1}{\pi}\left(\frac{1}{3}u^3 - u\right)\Big|_1^{-1}$$

$$= \frac{1}{\pi}\left[\left(\frac{-1}{3}+1\right)-\left(\frac{1}{3}-1\right)\right]$$

$$= \boxed{\frac{4}{3\pi} = f_{ave}}$$

E2 Is the average velocity the average of the instantaneous velocity?

$$V_{ave.} = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} V(t)\,dt$$

$$= \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \frac{dx}{dt}\,dt$$

$$= \frac{1}{t_2-t_1}\int_{x(t_1)}^{x(t_2)} dx$$

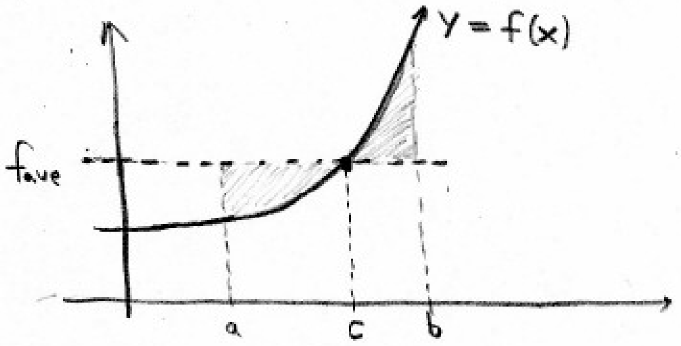
$$= \boxed{\frac{x(t_2)-x(t_1)}{t_2-t_1} = \frac{\Delta x}{\Delta t}}$$

Our new concept of averaging matches the old as it should.

Thm/ (MEAN VALUE THM for integrals) Let f be continous on $[a, b]$ then $\exists\, c \in [a, b]$ so that $f(c) = f_{ave}$. that is,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \xrightarrow{\text{aka}} (b-a)f(c) = \int_a^b f(x)\,dx$$

"Proof" by Picture:



graphically. $f_{ave} = y$ should have equal over / under estimates of the area. Meaning the shaded areas should be equal.

E3 If a ball has velocity $V = 30 - 9.8t$ when its thrown straight up what is the average velocity during the first 2 seconds? At what time does the instantaneous velocity match the average?

$$V_{ave} = \frac{1}{2}\int_0^2 (30 - 9.8t)\,dt$$

$$= \frac{1}{2}\left(30t - 4.9t^2\right|_0^2$$

$$= \frac{1}{2}\left(60 - 19.6\right)$$

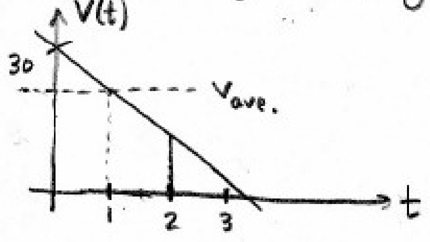
$$= \boxed{20.2}$$

$V_{ave} = V_{instantaneous}$ when $20.2 = 30 - 9.8t$

$$9.8t = 9.8$$

$$\boxed{t = 1}$$

Not to surprising if you consider the velocity-time graph



$\boxed{E4}$ (#15 pg. 470 §6.4) The voltage in U.S. houses typically varies sinusoidally from 155V to −155V with a frequency of 60 Hz. That is the voltage in your wall-socket as a function of time in seconds,

$$\mathcal{E}(t) = 155\sin(120\pi t)$$

a.) The average voltage is zero because, (average over 1 cycle)

$$\mathcal{E}_{ave.} = \frac{1}{\frac{1}{60}}\int_0^{1/60} 155\sin(120\pi t)\,dt$$

$$= \frac{60(155)}{120\pi}\left(-\cos(120\pi t)\right)\Big|_0^{1/60}$$

$$= \frac{60(155)}{120\pi}\left(-\cos(2\pi) + \cos(0)\right)$$

$$= \boxed{0 = \mathcal{E}_{ave.}}$$

b.) The voltage is not zero of course and is effectively like V_{rms} if we were to replace it with a constant voltage,

root mean square

$$\mathcal{E}_{rms} = \sqrt{\frac{1}{\frac{1}{60}}\int_0^{1/60}(\mathcal{E}(t))^2\,dt}$$

$$= \sqrt{(60)(155)^2\int_0^{1/60}\sin^2(120\pi t)\,dt}$$

$$= \sqrt{(60)(155)^2\int_0^{1/60}\frac{1}{2}(1-\cos(240\pi t))\,dt}$$

$$= \sqrt{(60)(155)^2\left[\frac{1}{2}\left(t - \frac{1}{240\pi}\sin(240\pi t)\right)\Big|_0^{1/60}\right]}$$

$$= \sqrt{(60)(155)^2\left[\frac{1}{2}\left(\frac{1}{60} - \frac{1}{240\pi}\sin(4\pi) - 0 + \frac{1}{240\pi}\sin(0)\right)\right]}$$

$$= \frac{155}{\sqrt{2}}$$

$$= \boxed{110\,V = \mathcal{E}_{rms}}$$

You could replace $\mathcal{E}(t)$ with a battery of $\mathcal{E} = 110V$ and deliver the same power to a given resistor.