The Binomial Series

There is a neat trick for calculating $(a+b)^k$. It's called Pascal's Triangle, I use it occasionally. Below I write the triangle and what the line $\Rightarrow$ for $(a+b)^k$.

$$
\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & \longrightarrow & (a+b)^0 = 1 \\
 & & & & 1 & & 1 & & & & & \longrightarrow & (a+b)^1 = a+b \\
 & & & 1 & & 2 & & 1 & & & & \longrightarrow & (a+b)^2 = a^2+2ab+b^2 \\
 & & 1 & & 3 & & 3 & & 1 & & & \longrightarrow & (a+b)^3 = a^3+3a^2b+3ab^2+b^3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & \longrightarrow & (a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & \longrightarrow & (a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{array}
$$

And so on, hopefully the pattern is clear. Well we can write this more compactly

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$$

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} = \frac{k!}{n!(k-n)!}$$

"k choose n", where $\binom{k}{0} = 1$

Binomial Th$^{\underline{m}}$ $(k \in \mathbb{N})$

This has been known for some time, however once k is allowed to be any real number we need an infinite series, to keep it simple we'll study $(1+x)^k$, once we know that we can easily find $(a+b)^k$ since $(a+b)^k = a^k(1+b/a)^k$.

$$
\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}
$$

Hence

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Therefore, assuming $R_n \to 0$ as $n \to \infty$,

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots}$$

Binomial Series

Remark: $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ converges for $|x| < 1$

While the I.O.C. depends on k, the result is

$$\text{I.O.C} = (-1, 1] \quad \text{if } -1 < k \le 0$$

$$\text{I.O.C} = [-1, 1] \quad \text{if } k \ge 0$$

$$\text{I.O.C} = (-1, 1) \quad \text{if } -1 > k$$

Proof left to reader. (I won't ask you to prove these)

$\boxed{E1}$ Expand $\frac{1}{(1+x)^2}$ using binomial series (this was $\boxed{E6}$ of §8.5)

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + \frac{(-2)(-2-1)}{2} x^2 + \frac{(-2)(-2-1)(-2-2)}{3!} x^3 + \dots$$

$$= \boxed{1 - 2x - 3x^2 + 4x^3 - \dots = \frac{1}{(1+x)^2}}$$

$\boxed{\text{for } x \in (-1, 1)}$

Same as $\boxed{E6}$ ☺

$\boxed{E2}$ $\frac{1}{\sqrt{1 - v^2/c^2}} = (1 - v^2/c^2)^{-1/2}$ let $u = -v^2/c^2$

$$= (1 + u)^{-1/2}$$

$$= 1 - \tfrac{1}{2}u + \frac{(-1/2)(-1/2 - 1)}{2} u^2 + \dots$$

$$= 1 - \tfrac{1}{2}u + \tfrac{3}{8}u^2 + \dots$$

$$= 1 + \tfrac{1}{2}\left(\tfrac{v}{c}\right)^2 + \tfrac{3}{8}\left(\tfrac{v}{c}\right)^4 + \dots$$

for $\left|\frac{v}{c}\right| < 1$ aka $\underline{-c < v < c}$

$\boxed{E3}$ $\frac{3}{1-x^2} = 3(1 - x^2)^{-1}$ $u = -x^2$

$$= 3(1+u)^{-1}$$

$$= 3\left[1 - u + \frac{(-1)(-1-1)}{2} u^2 + \dots\right]$$

$$= \boxed{3\left[1 + x^2 + x^4 + \dots\right] = \frac{3}{1-x^2}}$$

(with radius of convergence 1 $|x^2| < 1$)

- Alternatively you could have identified this to be a geometric series with $a = 3$ and $r = x^2$