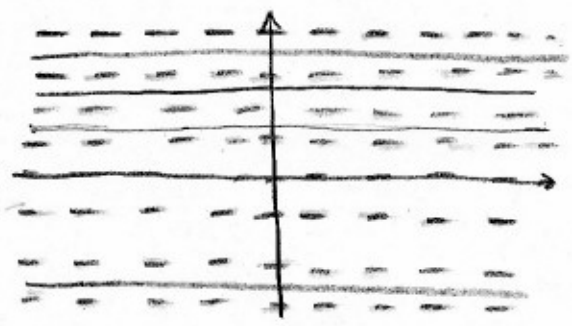


DIRECTION FIELDS

One method of understanding an ODE is to study what is called it's "Direction Field". Given an ODE we plot the tangents \Rightarrow by the ODE in the (xy)-plane, then given an initial starting point in the plane we can trace out a solⁿ.

E2 Let $y' = 0$ aka $\frac{dy}{dx} = 0$. The tangents are horizontal everywhere

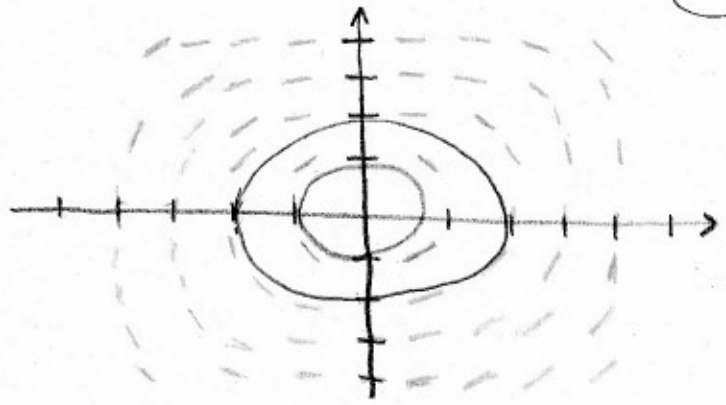


- the little blue dashes represent the tangents to the point.
- the colored lines are the graphs of solⁿ's to $y' = 0$, namely $y = c$

• notice the dashes indicate the slope near the dash (you don't have to draw the solⁿ's thru the dashes, you just need to be close to the same slope.)

E3 Let $y' = -\frac{x}{y}$

$$\begin{aligned}
 x^2 + y^2 = c &\Rightarrow 2x + 2y y' = 0 \\
 &\Rightarrow y' = -\frac{x}{y}
 \end{aligned}$$



\uparrow
this calculation reveals that solutions to the DE_qⁿ

$$\frac{dy}{dx} = -\frac{x}{y}$$

are circles. The nice thing about direction fields is even if you cannot solve a DE_qⁿ you may still deduce some features of the solⁿ by drawing solⁿ's into the direction field.

Defⁿ / An equilibrium solⁿ is one for which Y doesn't change; $Y' = 0$ along this solⁿ.

[E4] See problem 1. Let's analyze it algebraically and see how the direction field contains the same data.

$$Y' = Y(1 - \frac{1}{4}Y^2)$$

$$\text{Equilibrium Sol}^{\text{ns}} \stackrel{Y'=0}{\therefore} Y(1 - \frac{1}{4}Y^2) = 0 \Rightarrow \boxed{Y=0 \text{ or } Y=\pm 2}$$

this eqⁿ is autonomous, the direction field repeats at each X value since there was no X in the ODE, anyway we see these are the same curves as the direction field indicates for $Y'=0$ along whole curve.

[E5] See problem 5 $Y' = Y^2 - X^2$

$$Y' = 0 \Rightarrow Y^2 - X^2 = 0 \Rightarrow \boxed{Y = \pm X \text{ are equilibrium sol}^{\text{ns}}}$$

This corresponds to III.

[E6] See problem 3 $Y' = Y - 1$

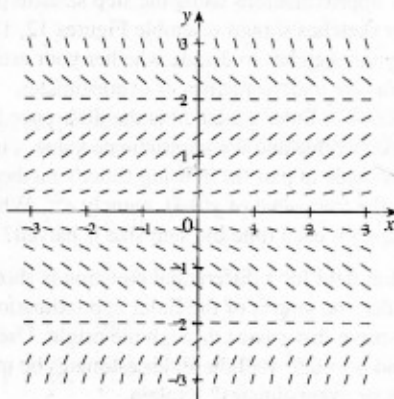
$$Y' = Y - 1 = 0 \Rightarrow \boxed{Y = 1 \text{ is equil. sol}^{\text{n}}}$$

this eqⁿ is autonomous, each X has same slopes repeated... (no X in ODE).

Attention : see pg. 168b for the problems mentioned/above.

Unfortunately, Stewart has changed some problems in the 2nd Ed. of the text, this is one of them, anyway the old problems are posted on the next page ↪.

1. A direction field for the differential equation $y' = y(1 - \frac{1}{4}y^2)$ is shown.
- (a) Sketch the graphs of the solutions that satisfy the given initial conditions.
- (i) $y(0) = 1$
 - (ii) $y(0) = -1$
 - (iii) $y(0) = -3$
 - (iv) $y(0) = 3$
- (b) Find all the equilibrium solutions.



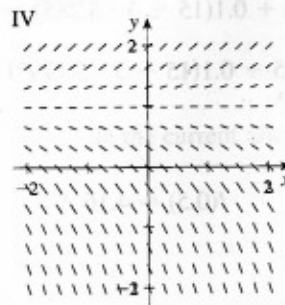
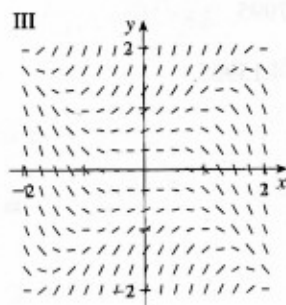
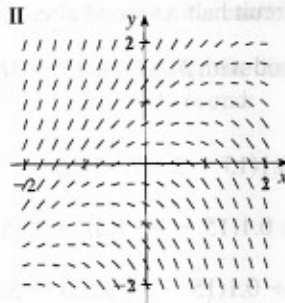
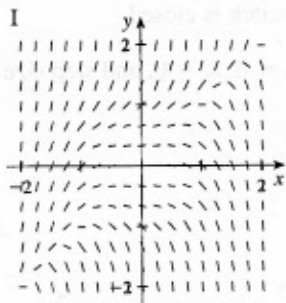
3-6 ■ Match the differential equation with its direction field (labeled I-IV). Give reasons for your answer.

3. $y' = y - 1$

4. $y' = y - x$

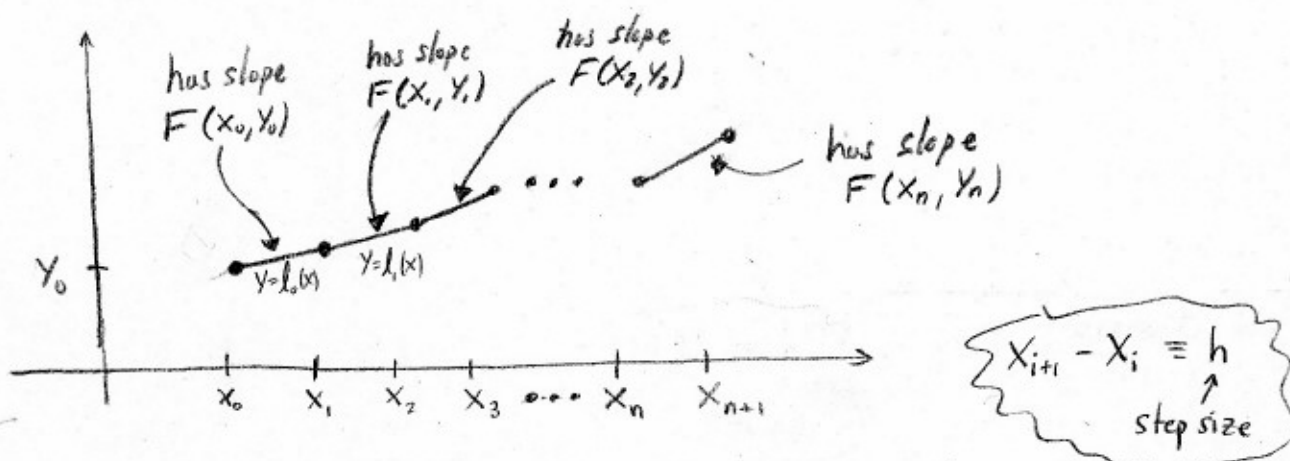
5. $y' = y^2 - x^2$

6. $y' = y^3 - x^3$



Euler's Method (Analytic application of Direction Field Concept.)

We can approx. a solⁿ to a 1st order ODE w/o solving it! Here's how
Given $Y' = F(x, y)$ and an initial condition $Y(x_0) = Y_0$.



We approx. the solⁿ with a line from x_0 to x_1 with slope $F(x_0, y_0)$, what's the eqⁿ for that line? Call it $l_0(x) = mx + b$

i) $m = F(x_0, y_0)$
 ii) $l(x_0) = y_0 = F(x_0, y_0) \cdot x_0 + b$
 $\Rightarrow b = y_0 - F(x_0, y_0) \cdot x_0$
 $\therefore l_0(x) = F(x_0, y_0) \cdot x + y_0 - F(x_0, y_0) \cdot x_0$
 aka $l_0(x) = y_0 + F(x_0, y_0)(x - x_0)$

How do we continue, well we need (x_1, y_1) where $x_1 = x_0 + h$ and y_1 comes from the last link,

$y_1 = l_0(x_1)$
 $= y_0 + F(x_0, y_0)(x_1 - x_0)$
 $= y_0 + h F(x_0, y_0) = y_1$

$\therefore l_1(x) = y_1 + F(x_1, y_1)(x - x_1)$

And so on, this is why the book writes, (see Ex. 3 pg. 509)

$y_1 = y_0 + h F(x_0, y_0)$
 $y_2 = y_1 + h F(x_1, y_1)$
 \vdots
 $y_n = y_{n-1} + h F(x_{n-1}, y_{n-1})$

Euler's Method

E7 $Y' = Y + XY$ so $F(x,y) = Y'$ where $F(x,y) = Y + XY$.

$Y(0) = 1 \Rightarrow X_0 = 0 \neq Y_0 = 1$. The step size $h = 0.1$
And we want to find $Y(0.5) \cong Y_5$.

$$Y_1 = Y_0 + h F(X_0, Y_0) = 1 + (0.1)[1 + 0.1] = \underline{1.1} = Y_1$$

$$Y_2 = Y_1 + h F(X_1, Y_1) = 1.1 + (0.1)[1.1 + 0.1(1.1)] = \underline{1.221} = Y_2$$

$$Y_3 = Y_2 + h F(X_2, Y_2) = 1.221 + (0.1)[1.221 + (0.2)(1.221)] = \underline{1.368} = Y_3$$

$$Y_4 = Y_3 + h F(X_3, Y_3) = 1.368 + (0.1)[1.368 + (0.3)(1.368)] = \underline{1.546} = Y_4$$

$$Y_5 = Y_4 + h F(X_4, Y_4) = 1.546 + (0.1)[1.546 + (0.4)(1.546)] = \underline{1.762} = Y_5$$

Hence $Y(0.5) \cong 1.762$ # 23 in text.

Remark: Euler's Method is nothing more than trying to trace out solⁿ's in a direction field. Think about it...