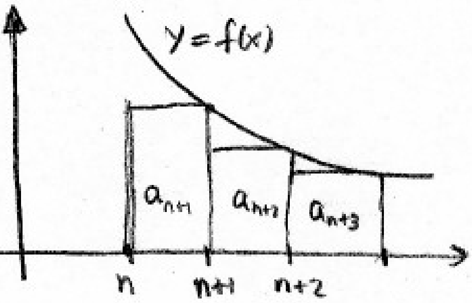
Estimating the Sum of a Series (§ 8.3 & § 8.4)

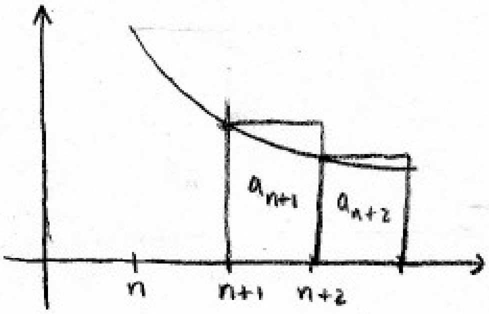
Usually an analytic sol^{n} of $s = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}(S_n)$ is not possible. In such cases we are forced to truncate s to S_n the question arises then, <u>how close to the real s are we with S_n</u>? We can quantify this by the remainder. (also called the error)

$$\boxed{R_n \equiv S - S_n = a_{n+1} + a_{n+2} + \cdots = \sum_{k=n+1}^{\infty} a_k}$$

In this lecture we will learn how to find bounds on R_n.

Motivational Picture: Let $f(n) = a_n$ & smoothly continue f between those points.





$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^{\infty} f(x)\,dx \qquad\qquad R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx$$

Th^{m}/ (Error Estimate via Integral Test). If $\sum a_n$ converges by the integral test and $R_n = S - S_n$ we have for $f(n) = a_n$,

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$$

$\boxed{E1}$ $s = \sum_{n=1}^{\infty} \frac{1}{n^2}$ approx. s by S_5, find a bound on the error in doing this.

Of course $f(x) = \frac{1}{x^2}$ is decreasing, positive fnct with $f(n) = \frac{1}{n^2}$ which converges $\int_1^{\infty} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{\infty} = 1$. Notice then

$$\int_{n+1}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n+1} \quad \text{and} \quad \int_n^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n}$$

Thus by E.E. via I.T Th^{m} $\frac{1}{n+1} \leq R_n \leq \frac{1}{n}$ that is for S_5 we have $n = 5$ so $\frac{1}{6} \leq \text{error} \leq \frac{1}{5}$.

Recall $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.645$ (Beyond our Course, I just quote result)

While $S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{5269}{3600} = 1.464 \therefore R_n = 1.645 - 1.464 = 0.181$

$$0.1\overline{6} \leq R_n = 0.181 \leq 0.2$$

Corollary to ERROR EST. VIA INT. TEST Thm: We can better approximate s since a direct implication of Thm is

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_n^{\infty} f(x)\,dx$$

$\boxed{E2}$ How does this work for $\boxed{E1}$? We'll we found $s_5 = 1.464$ while $\int_{n+1}^{\infty} f(x)\,dx = \frac{1}{n+1}$ & $\int_n^{\infty} f(x)\,dx = \frac{1}{n}$ thus for $n = 5$ we get

$$1.464 + \frac{1}{6} \le s \le 1.464 + \frac{1}{5}$$

$$1.631 \le s \le 1.664$$

where $s = \frac{\pi^2}{6} \cong 1.64493$ actually

What can we conclude, $s \cong 1.6$ ← these digits are certain. If we let $n = 20$ we could do better,

$$s_{20} = \sum_{n=1}^{20} \frac{1}{n^2} = 1.59616$$

Apply cor. to Thm.

$$1.59616 + \frac{1}{21} \le s \le 1.59616 + \frac{1}{20}$$

$$1.64378 \le s \le 1.64616$$

$s \cong 1.64$

$n = 20$ gives us 3 digits for sure.

Th^{m}/(ALTERNATING SERIES ESTIMATION Th^{m}) If $s = \sum (-1)^{n-1} b_n$ satisfies
(a) $0 < b_{n+1} \leq b_n$
(b) $\lim_{n\to\infty} b_n = 0$
Then $|R_n| = |s - s_n| \leq b_{n+1}$. In other words if an alternating series converges then s_n has an error smaller than the next term b_{n+1}

$\boxed{E3}$ Find $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to 3 decimal places. $b_n = \frac{1}{n!}$

$$0 \leq \frac{1}{(n+1)!} = \frac{1}{(n+1)n!} \leq \frac{1}{n!} \quad \therefore 0 \leq b_{n+1} \leq b_n$$

Additionally $\frac{1}{n!} \leq \frac{1}{n}$ and $\frac{1}{n} \to 0$ as $n \to \infty$ $\therefore \frac{1}{n!} \to 0$ as $n \to \infty$.

Notice that

$b_0 = 1$, $b_1 = 1$ $b_2 = 0.5$, $b_3 = 0.1\overline{6}$, $b_4 = 0.41667$, $b_5 = 0.008333$.

$b_6 = 0.001389$, $\underbrace{b_7 = 0.000198}_{\text{this is error in } S_6}$

$\therefore$ S_6 will do.

$S_6 = 0.368056$ with error $b_7 = 0.000198$

$\therefore$ $\boxed{S \approx 0.368}$

Remark: there are about 5 homework problems that utilize this very useful Th^{m}. Whenever we want a bound on the error from something that has an alternating series representation.
pg. 604 # 25 & 28 for example.