2^{nd} Order Linear Differential Eq^n's with constant coefficients

Now that we have studied 1^{st} order ODE's in some depth we'll study the simplest class of 2^{nd} order ODE's. Let us begin with the easiest case called "homogeneous"

$$\boxed{Ay'' + By' + Cy = 0} \quad (*)$$

What is a sol^n? Well let's make an ansatz, that is an educated guess; $y = e^{\lambda x}$ if this is the sol^n then note

$$y' = \lambda e^{\lambda x}$$
$$y'' = \lambda^2 e^{\lambda x}$$

Substituting yields:

$$A\lambda^2 e^{\lambda x} + B\lambda e^{\lambda x} + Ce^{\lambda} = (A\lambda^2 + B\lambda + C)e^{\lambda} = 0$$

Now since $e^{\lambda} \neq 0$ for any $\lambda \in \mathbb{C}$ we find the Characteristic Eq^n for $(*)$ namely

$$\boxed{A\lambda^2 + B\lambda + C = 0} \quad \text{Ch.Eq}^n$$

Notice that we can allways solve this eq^n $(A \neq 0)$ with the quadratic formula:

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Where $B^2 - 4AC$ is the discriminant which discriminates which type of sol^n we get. Lets label the cases

I.) $B^2 - 4AC > 0$ distinct real roots

II.) $B^2 - 4AC = 0$ repeated real roots

III.) $B^2 - 4AC < 0$ Complex roots

With this in mind now we'll solve $(*)$ in each of the above cases. Notice since $(*)$ is 2^{nd} order we need TWO arbitrary $\int$ constants $\therefore$ TWO "linearly independent" sol^n's, happily quad. eq^n's give us TWO sol^n's. We began looking for a sol^n and found the general sol^n! neat.

I.) $B^2-4AC > 0$: DISTINCT REAL ROOTS

$$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2-4AC}}{2A} \in \mathbb{R}$$

The sol^{n} is simply $\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$
since $\lambda_1, \lambda_2 \in \mathbb{R}$ these are just plain-old exponentials.

$\boxed{E1}$ $y'' + 3y' + 2y = 0 \Rightarrow \lambda^2 + 3\lambda + 2 = 0$

Thus $(\lambda+1)(\lambda+2) = 0 \therefore \underline{\lambda_1 = -1 \text{ and } \lambda_2 = -2}$ Sol^{n}'s to Ch. Eq^{n}.
(No need to use quad. eq^{n} if it factors, remember ?)
Hence $\boxed{y = c_1 e^{-x} + c_2 e^{-2x}}$ where c_1, c_2 are arbitrary real constants.

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II.) $B^2-4AC = 0$: REPEATED ROOTS

$$\lambda_{1,2} = -B/2A \equiv \lambda$$

The sol^{n} is $\boxed{y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}}$. I'll not try to derive where the x comes from, rather we'll show it works, $y_2 \equiv x e^{\lambda x}$

$$\frac{d}{dx}\left(xe^{\lambda x}\right) = e^{\lambda x} + \lambda x e^{\lambda x}$$

$$\frac{d^2}{dx^2}\left(xe^{\lambda x}\right) = \lambda e^{\lambda x} + \lambda e^{\lambda x} + x\lambda^2 e^{\lambda x}$$

$$\begin{aligned}
Ay_2'' + By_2' + Cy_2 &= A\left(2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}\right) + B\left(e^{\lambda x} + \lambda x e^{\lambda x}\right) + Cxe^{\lambda x} \\
&= A\left(\frac{-2B}{2A}e^{\lambda x} + \frac{B^2}{4A^2}xe^{\lambda x}\right) + B\left(e^{\lambda x} - \frac{B}{2A}xe^{\lambda x}\right) + Cxe^{\lambda x} \\
&= \cancel{-Be^{\lambda x}} + \cancel{\frac{B^2}{4A}xe^{\lambda x}} + \cancel{Be^{\lambda x}} - \cancel{\frac{B^2}{2A}xe^{\lambda x}} + \cancel{\frac{B^2}{4A}xe^{\lambda x}} \\
&= 0
\end{aligned}$$

Hence $y_1 = e^{\lambda x}$ satisfies (*) clearly, and $y_2 = xe^{\lambda x}$ satisfies (*) by above calculation. So also $y = c_1 y_1 + c_2 y_2$ will solve (*) since differentiation is a linear operation.

E2 $y'' - 2y' + y = 0$

$\Rightarrow \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \quad \therefore \underline{\lambda = 1}$

$\therefore \boxed{y = c_1 e^x + c_2 x e^x}$

III. $B^2 - 4AC < 0$: Complex Roots

Since it is customary to use explicitly real-valued fcts. we need to understand what $e^{\lambda x}$ means when $\lambda \in \mathbb{C}$.

$\boxed{\text{Th}^{\text{m}} \text{ (Euler's Id.) } e^{i\theta} = \cos\theta + i\sin\theta.}$

Additional Properties: these follow easily from Th$^{\text{m}}$, and (4) is a definition

(1) $e^{-i\theta} = \cos\theta - i\sin\theta$

(2) $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

(3) $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

(4) $e^{(a+ib)x} \equiv e^{ax}e^{ibx} = e^{ax}(\cos(bx) + i\sin(bx))$

With these facts in-hand lets derive the real-form of type III. sol$^{\text{n}}$'s. As in case I $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

$$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-B}{2A} \pm \frac{\sqrt{(-1)}\sqrt{4AC - B^2}}{2A}$$

$$= \frac{-B}{2A} \pm i\frac{\sqrt{4AC - B^2}}{2A}$$

$$= \alpha \pm i\beta$$

Where we defined $\alpha \equiv \frac{-B}{2A}$ \& $\beta \equiv \frac{\sqrt{4AC - B^2}}{2A}$

Notice $\alpha, \beta \in \mathbb{R}$.

III. (Complex Roots Continued) Remark: We want a real sol^{n}.

So we may write $e^{\lambda_1 x} = e^{(\alpha + i\beta)x}$ and $e^{\lambda_2 x} = e^{(\alpha - i\beta)x}$

Notice $\tilde{y} = Re\{\tilde{y}\} + i\, Im\{\tilde{y}\}$, its a complex-valued fnct. Lets find $Re\{\tilde{y}\}$

$$
\begin{aligned}
\tilde{y} &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
&= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \\
&= c_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + c_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) \\
&= e^{\alpha x}\left[(c_1 + c_2)\cos(\beta x) + i(c_1 - c_2)\sin\beta x\right] \\
&= e^{\alpha x}\left[\tilde{c}_1 \cos(\beta x) + \tilde{c}_2 \sin(\beta x)\right]
\end{aligned}
$$

Where $\tilde{c}_1 \equiv c_1 + c_2$ and $\tilde{c}_2 = i(c_1 - c_2)$. Let $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$.

$$
\begin{aligned}
\tilde{c}_1 &= a_1 + ib_1 + a_2 + ib_2 = a_1 + a_2 + i(b_1 + b_2) \\
\tilde{c}_2 &= i(a_1 + ib_1 - a_2 - ib_2) = (b_2 - b_1) + i(a_1 - a_2)
\end{aligned}
$$

Thus taking the real-part of $\tilde{y}$ gives

means to take the real part

$$y = e^{\alpha x}\left[(a_1 + a_2)\cos(\beta x) + (b_2 - b_1)\sin(\beta x)\right] = Re\{\tilde{y}\}$$

We study real-valued sol^{n}'s so we'll allways use $Re\{\tilde{y}\} = y$

$$\boxed{y = e^{\alpha x}\left[a\cos(\beta x) + b\sin(\beta x)\right]}$$ general sol^{n} to case III.

Remark: The calculation we did above is typical of complex analysis, we goto the complex generalization then do the calculation where it's easy and in the end take the $Re\{\ \}$ part.

E3 $y'' + y = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \therefore \alpha = 0, \beta = 1$

Hence $\boxed{y = a\cos(x) + b\sin(x)}$

More examples of homogeneous eq$^{\underline{ns}}$ $Ay''+By'+Cy=0$

E4 $2y''+6y'-11y=0$

$2\lambda^2+6\lambda-11=0 \Rightarrow \lambda = \dfrac{-6\pm\sqrt{36+4(2)(11)}}{4} = \dfrac{-6\pm\sqrt{124}}{4}$

Thus $\lambda = \dfrac{-3\pm\sqrt{31}}{2}$ which is distinct real case, (I.)

$$\boxed{y = c_1 e^{\frac{-3+\sqrt{31}}{2}x} + c_2 e^{\frac{-3-\sqrt{31}}{2}x}}$$

E5 $\dfrac{d^2r}{d\theta^2}+16r=0$

$\lambda^2+16=0 \Rightarrow \lambda = \pm 4i$ or $\alpha=0$ and $\beta=4$

$$\boxed{r(\theta) = c_1\cos(4\theta)+c_2\sin(4\theta)}$$

E6 $\varphi''+2\varphi'+5\varphi=0$ where $\varphi' \equiv \dfrac{d\varphi}{dt}$.

$\lambda^2+2\lambda+5=0$

$\lambda = \dfrac{-2\pm\sqrt{4-4(5)}}{2} = -1\pm\dfrac{\sqrt{-16}}{2} = -1\pm 2i$, $\alpha=-1$ & $\beta=2$

$$\boxed{\varphi(t) = e^{-t}\left(a\cos(2t)+b\sin(2t)\right)}$$

E7 $\psi''-16\psi=0$ where $\dfrac{d\psi}{dx} \equiv \psi'$,

$\lambda^2-16=0 \Rightarrow \underline{\lambda=\pm 4}$

$$\boxed{\psi = c_1e^{4x}+c_2e^{-4x}}$$

__Remark:__ There is a prettier way to phrase this case $(B=0)$ using the hyperbolic trig fncts.

$\cosh(x) = \frac{1}{2}(e^x+e^{-x})$ & $\sinh(x) = \frac{1}{2}(e^x-e^{-x})$

$\psi(x) = A\cosh(4x)+B\sinh(4x)$ ← equivalent sol$^{\underline{n}}$ to

$= \frac{A}{2}(e^{4x}+e^{-4x}) + \frac{B}{2}(e^{4x}-e^{-4x})$

$= \left(\frac{A}{2}+\frac{B}{2}\right)e^{4x} + \left(\frac{A}{2}-\frac{B}{2}\right)e^{-4x}$

$= c_1e^{4x}+c_2e^{-4x}$