Improper Integrals

Improper integrals involve unbounded functions and/or infinite intervals of integration. We begin with those which involve infinite intervals,

Def^n/ Assuming the limits below exist,

a) $\int_a^\infty f(x)\,dx \equiv \lim_{t\to\infty} \int_a^t f(x)\,dx$

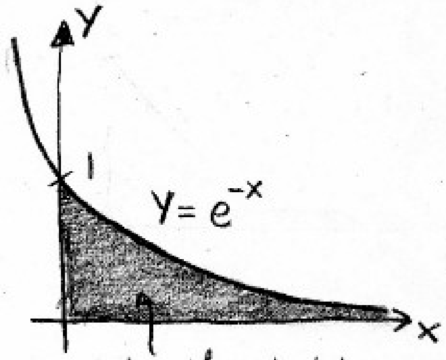
b) $\int_{-\infty}^b f(x)\,dx \equiv \lim_{t\to-\infty} \int_t^b f(x)\,dx$

c.) $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$ (usually use $a=0$)

These are all convergent integrals, when these limits exist. When the limits d.n.e we say the the integrals are divergent.

$\boxed{E1}$

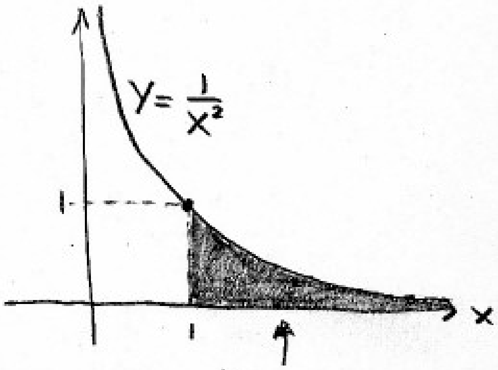
$$\begin{aligned}\int_0^\infty e^{-x}\,dx &\equiv \lim_{t\to\infty}\int_0^t e^{-x}\,dx\\ &= \lim_{t\to\infty}\left(-e^{-x}\Big|_0^t\right)\\ &= \lim_{t\to\infty}\left(\underbrace{-e^{-t}}_{\to 0}+1\right)\\ &= \boxed{1}\end{aligned}$$



integral calculates the pictured area.

$\boxed{E2}$

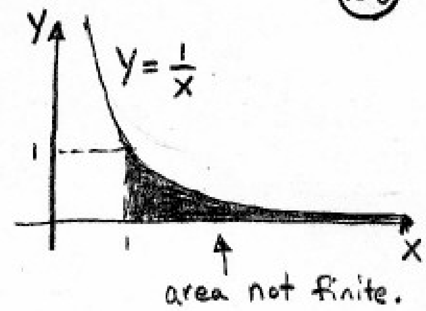
$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &\equiv \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx\\ &= \lim_{t\to\infty}\left(\frac{-1}{x}\Big|_1^t\right)\\ &= \lim_{t\to\infty}\left(\underbrace{\frac{-1}{t}}_{\to 0}+1\right)\\ &= \boxed{1}\end{aligned}$$



this area is apparently 1.

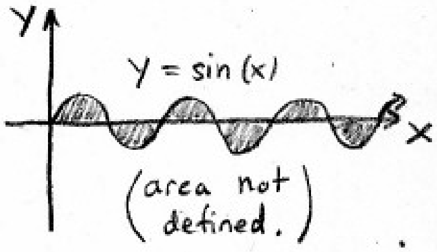
E3

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$$
$$= \lim_{t\to\infty}\left(\ln(x)\Big|_1^t\right)$$
$$= \lim_{t\to\infty}\left(\ln(t)\right)$$
$$= \infty \quad (\text{divergent})$$



E4

$$\int_{-\infty}^0 \sin(x)\,dx = \lim_{t\to-\infty}\int_t^0 \sin(x)\,dx$$
$$= \lim_{t\to-\infty}\left(-\cos(x)\Big|_t^0\right)$$
$$= \lim_{t\to-\infty}\left(-1+\cos(t)\right)$$
$$= \text{d.n.e} \quad (\text{divergent because } \cos(t) \text{ "oscilates at } \infty\text{"})$$



E5

$$\int_0^\infty \frac{2x}{1+x^4}\,dx = \lim_{t\to\infty}\left(\int_0^t \frac{2x}{1+x^4}\,dx\right)$$
$$= \lim_{t\to\infty}\left(\int_0^{t^2}\frac{du}{1+u^2}\right)$$

$u = x^2$, $du = 2x\,dx$, $u(0) = 0$, $u(t) = t^2$

$$= \lim_{t\to\infty}\left(\tan^{-1}(u)\Big|_0^{t^2}\right)$$
$$= \lim_{t\to\infty}\left(\tan^{-1}(t^2) - \overset{0}{\cancel{\tan^{-1}(0)}}\right)$$
$$= \boxed{\pi/2}$$

- Reading Suggested (Examples 2, 3 & 4 in text pg. 425–426)
- Also you might review limits involving ∞ from calculus I.

$\boxed{E6}$

$$\int x^2 e^{-x^3}dx = \int \frac{-1}{3} e^u\,du \quad \leftarrow \boxed{\begin{aligned}u &= -x^3\\ du &= -3x^2dx\end{aligned}}$$

$$= \frac{-1}{3}e^u + C$$

$$= \frac{-1}{3}e^{-x^3} + C$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^3}dx = \lim_{t_1\to-\infty}\int_{t_1}^{0} x^2e^{-x^3}dx + \lim_{t_2\to\infty}\int_0^{t_2} x^2e^{-x^3}dx$$

$$= \lim_{t_1\to-\infty}\left(\frac{-1}{3}e^{-x^3}\Big|_{t_1}^{0}\right) + \lim_{t_2\to\infty}\left(\frac{-1}{3}e^{-x^3}\Big|_0^{t_2}\right)$$

$$= \lim_{t_1\to-\infty}\left(\frac{-1}{3} + \frac{1}{3}\underbrace{e^{-t_1^3}}_{\infty}\right) + \lim_{t_2\to\infty}\left(\frac{-1}{3}\underbrace{e^{-t_2^3}}_{0} + \frac{1}{3}\right)$$

= divergent

$\boxed{E7}$

$$\int \frac{\ln(x)}{x^3}dx = \frac{-\ln(x)}{2x^2} - \int \frac{-1}{2x^2}\frac{dx}{x}$$

$u = \ln(x)$	$dV = dx/x^3$
$du = \frac{dx}{x}$	$V = \frac{-1}{2x^2}$

$$= \frac{-1}{2x^2}\ln(x) + \frac{1}{2}\frac{-1}{2x^2} + C$$

$$= \frac{-1}{2x^2}\left(\ln(x) + \frac{1}{2}\right) + C$$

$$\int_1^{\infty}\frac{\ln(x)}{x^3}dx = \lim_{t\to\infty}\int_1^t \frac{\ln(x)\,dx}{x^3}$$

$$= \lim_{t\to\infty}\left(\frac{-1}{2x^2}\left(\ln(x)+\frac{1}{2}\right)\Big|_1^t\right)$$

$$= \lim_{t\to\infty}\left(\frac{\ln(t)+1/2}{-2t^2} + \frac{\ln(1)+1/2}{2}\right)$$

$$\overset{\mathcal{L}}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{t\to\infty}\left(\frac{1/t}{-4t}\right) + \frac{1}{4} \leftarrow$$ pulled this out before doing L-Hopital's Rule on the 1st term.

$$= \lim_{t\to\infty}\underbrace{\cancel{\left(\frac{-1}{4t^2}\right)}}_{0} + \frac{1}{4}$$

$$= \boxed{1/4}$$

Improper Integrals Part II (§5.10)

We now deal with the problem of calculating the area under a curve at a vertical asymptote. Sometimes it can be finite see $\boxed{E10}$.

Def^n/ Provide the limits below exist (are real numbers)

a.) $$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx \qquad (b \notin dom(f))$$

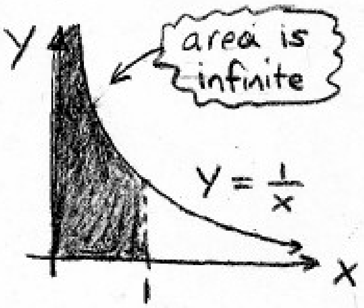
b.) $$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx \qquad (a \notin dom(f))$$

c.) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad (c \notin dom(f))$$

If the limit in a, b or c d.n.e we say the integrals diverge.

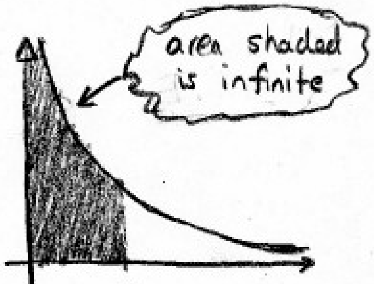
$\boxed{E8}$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x}\,dx$$
$$= \lim_{t \to 0^+} \left(\ln(x)\Big|_t^1\right)$$
$$= \lim_{t \to 0^+} \left(\underbrace{\ln(1)}_{0} - \underbrace{\ln(t)}_{\infty}\right) = \boxed{\infty} \text{ divergent.}$$



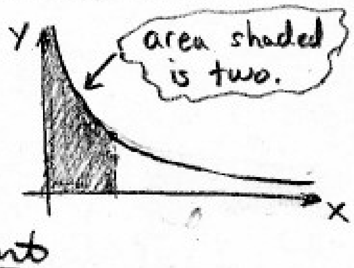
$\boxed{E9}$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x^2}\,dx$$
$$= \lim_{t \to 0^+} \frac{-1}{x}\Big|_t^1$$
$$= \lim_{t \to 0^+} \left(-1 + \underbrace{\frac{1}{t}}_{\infty}\right) = \boxed{\infty} \text{ divergent}$$



$\boxed{E10}$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$$
$$= \lim_{t \to 0^+} \left(2\sqrt{x}\Big|_t^1\right)$$
$$= \lim_{t \to 0^+} \left(2 - 2\sqrt{t}\right) = \boxed{2} \text{ convergent}$$



E11

$$\int_0^{\pi/4} \csc^2(x)\,dx = \lim_{t\to 0^+} \int_t^{\pi/4} \csc^2(x)\,dx$$

$$= \lim_{t\to 0^+} \left[-\cot(x) \Big|_t^{\pi/4} \right]$$

$$= \lim_{t\to 0^+} \left[-\underbrace{\cot(\pi/4)}_{1} + \underbrace{\cot(t)}_{\infty} \right] \quad \underline{\text{diverges.}}$$

E12

$$\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx = \lim_{t\to 9^-} \int_1^t \frac{1}{\sqrt[3]{x-9}}\,dx$$

$$= \lim_{t\to 9^-} \left(\frac{3}{2}(x-9)^{2/3} \Big|_1^t \right)$$

$$= \lim_{t\to 9^-} \left(\frac{3}{2}(\underbrace{t-9}_{0})^{2/3} - \frac{3}{2}(-8)^{2/3} \right)$$

$$= \boxed{-6}$$

E13

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}}$$

BOTH TYPES OF IMPROPREITY HERE. (V.A. at $x=2$)

$$\int \frac{dx}{x\sqrt{x^2-4}} = \sec^{-1}\left(\tfrac{1}{2}x\right) + C$$

follows from the trig-subst. $x = 2\sec\theta$. it takes a little work, try it.

$$\int_2^3 \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+} \int_t^3 \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+} \left(\sec^{-1}\left(\tfrac{3}{2}\right) - \sec^{-1}\left(\tfrac{t}{2}\right) \right)$$

$$= \boxed{\sec^{-1}(3/2) - \underbrace{\sec^{-1}(1)}_{0}}$$

$$\int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to\infty} \left(\sec^{-1}\left(\tfrac{1}{2}x\right) \Big|_3^t \right)$$

$$= \lim_{t\to\infty} \left(\sec^{-1}\left(\tfrac{t}{2}\right) - \sec^{-1}(3/2) \right)$$

$$= \boxed{\pi/2 - \sec^{-1}(3/2)}$$

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \cancel{\sec^{-1}(3/2)} + \frac{\pi}{2} - \cancel{\sec^{-1}(3/2)} = \boxed{\frac{\pi}{2}}$$