The Logistic Eq^{n}

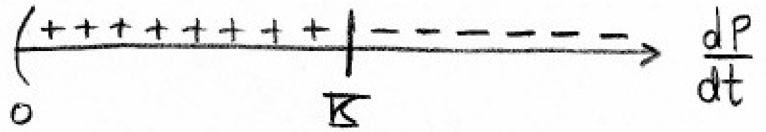
This is another model for population growth, the basic idea is that when the population P is small then $\frac{dP}{dt} = kP$ but as P gets big the resources are all used up and the population is unable to continue growing past some limiting population K = the carrying capacity. The simplest eq^{n} encoorporating the above features is

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \quad \text{The Logistic Eq}^{n}}$$

Notice that as $P \to K$ we have $\frac{dP}{dt} \to 0$. As we desired the growth slows to zero as we approach the carrying capacity. Additionally when $P \ll K$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \cong kP$$

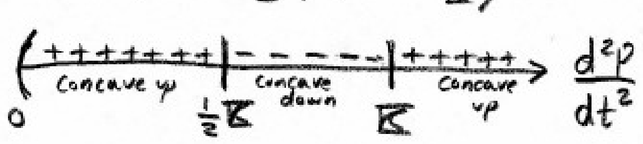
So for small population this model is like exponential growth. Now lets figure out what general features the sol^{n}'s to the Logistic Eq^{n} must have, (time for some calc. I)



P increases when $P < K$
P decreases when $P > K$

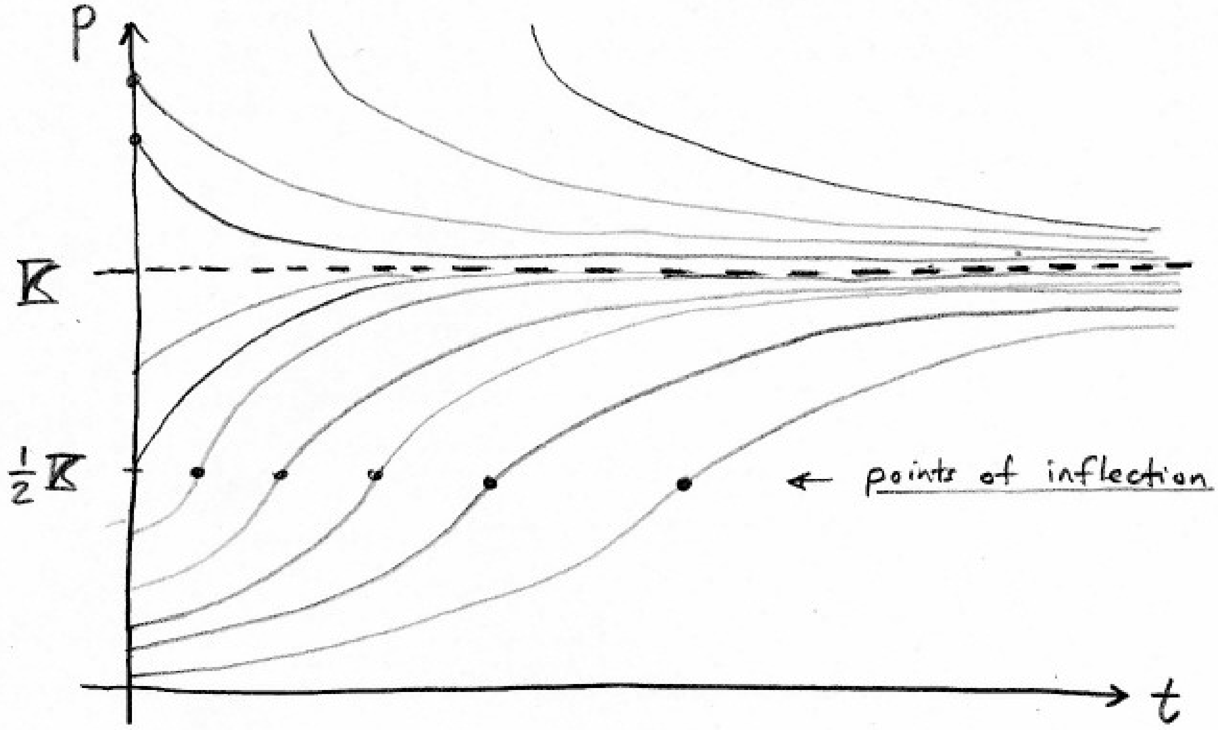
What about concavity? Lets differentiate,

$$\begin{aligned}\frac{d^2P}{dt^2} &= k\frac{dP}{dt}\left(1 - \frac{P}{K}\right) - \frac{k}{K}P\frac{dP}{dt} \\ &= k\left(1 - \frac{2P}{K}\right)\frac{dP}{dt} \\ &= k^2\left(1 - \frac{2P}{K}\right)\left(1 - \frac{P}{K}\right)\end{aligned}$$



Notice $\frac{dP}{dt}$ is maximized at $P = \frac{1}{2}K$.

Graph of Sol^n's to Logistic Eq^n



Inevitably as $t \to \infty$ the sol^n goes to K no matter what the intial condition was.

Remark: We have yet to find a sol^n. Next we'll explicitly solve the Log. Eq^n. I think its interesting we can see so much just from studying the DEq^n directly.

Logistic Eqⁿ Analytic Solⁿ

$$\frac{dP}{dt} = kP\left(1-\frac{P}{K}\right) \Rightarrow \frac{dP}{P(1-P/K)} = k\,dt$$

Now to integrate in P we'll use partial fractions,

$$\frac{1}{P(1-P/K)} = \frac{A}{P} + \frac{B}{1-P/K}$$

$$1 = A(1-P/K) + BP \quad \xrightarrow{P=0} A = 1 \qquad \xrightarrow{P=K} 1 = BK \therefore B = 1/K$$

Thus $\dfrac{1}{P(1-P/K)} = \dfrac{1}{P} + \dfrac{1}{K-P}$

$$\int \frac{dP}{P(1-P/K)} = \int\left(\frac{1}{P} + \frac{1}{K-P}\right)dP = \ln|P| - \ln|K-P| = \ln\left|\frac{P}{K-P}\right|$$

$$\int k\,dt = kt + C$$

Hence $\ln\left|\dfrac{P}{K-P}\right| = kt + C \Rightarrow \left|\dfrac{P}{K-P}\right| = e^{c}e^{kt} \Rightarrow \dfrac{P}{K-P} = Ae^{kt}$

$A = \pm e^{c}$

Now solve for P

$$P = (K-P)Ae^{kt}$$

$$P(1+Ae^{kt}) = AKe^{kt} \Rightarrow P = \frac{AKe^{kt}}{1+Ae^{kt}} = \boxed{\frac{K}{1+Ae^{-kt}} = P(t)}$$

Exercise: Verify for yourself that the conclusions we reached for inc/dec concave up/down etc... are duplicated by this solⁿ.

Remark: What ever the intial population is the final population is K

$$\lim_{t\to\infty}\left(\frac{K}{1+\underbrace{Ae^{-kt}}_{\to 0}}\right) = K$$

$\boxed{\text{E1}}$ Suppose that $\frac{dP}{dt} = 0.05P - 0.0005P^2$

Then what is the carrying capacity K ? and k ?

$$\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{100}\right) = kP\left(1 - \frac{P}{K}\right)$$

Comparing we identify $\boxed{K = 100 \text{ and } k = 0.05}$

$\boxed{\text{E2}}$ Suppose the carrying capacity of the US is 1000 (million). Additionally in 1990 $P = 250$ and in 2000 $P = 275$ million. Find $P(t)$ then predict the pop. in 2010 and 2100.

$$P(t) = \frac{1000}{1 + Ae^{-kt}}$$

Let 1990 be $t = 0$, then $P(0) = \frac{1000}{1+A} = 250 \Rightarrow \boxed{A = 3}$

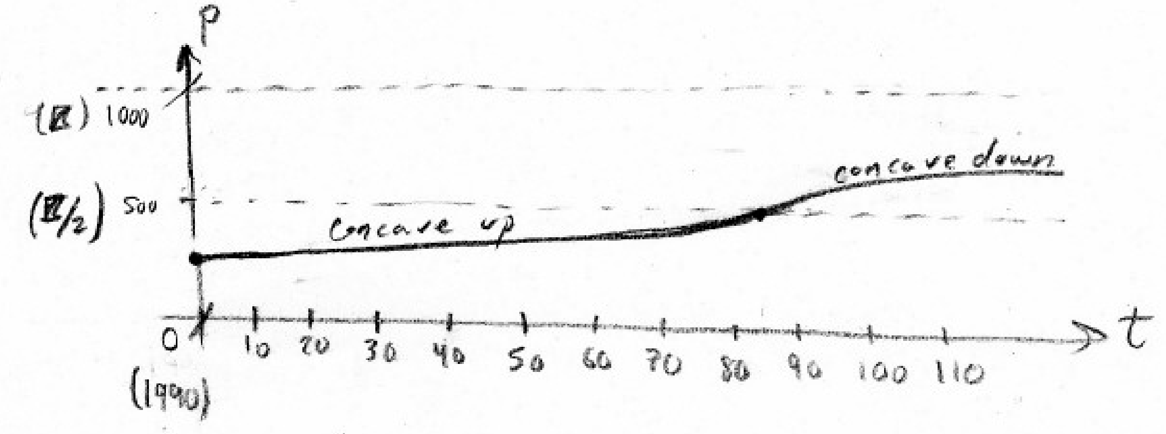
Additionally: $P(10) = \frac{1000}{1+3e^{-10k}} = 275 \Rightarrow 725 = 275(3e^{-10k})$

$$\Rightarrow \frac{725}{3 \cdot 275} = e^{-10k} = \frac{29}{33}$$

$$\Rightarrow \boxed{k = \frac{\ln(29/33)}{-10} = 0.01292}$$

$$P(20) = \frac{1000}{1+e^{-0.13(20)}} = 301 \text{ million in } 2010$$

$$P(110) = \frac{1000}{1+e^{-0.13(110)}} = 580 \text{ million in } 2100$$



$P(t) = \frac{K}{2} = \frac{K}{1+3e^{-kt}} \Rightarrow 2 = 1 + 3e^{-kt}$

$\frac{1}{3} = e^{-kt}$

$$\Rightarrow \boxed{t = \frac{\ln(3)}{k} = \frac{\ln(3)}{0.01292} = 85 \Rightarrow P(85) = K/2}$$