NUMERICAL INTEGRATION : AN INTRODUCTION :

Questions : ① If we use L_n, R_n, M_n then how close is our aprox. to the real answer $\int_a^b f(x)dx \equiv I$

② Are there any other approximation shemes ?

③ How do the different approx's compare (usually)

Let's begin with question ②

Trapezoidal Rule with n-trapezoids : $T_n \cong \int_a^b f(x)\,dx$

$$\boxed{T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \qquad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x}$$

"Proof"

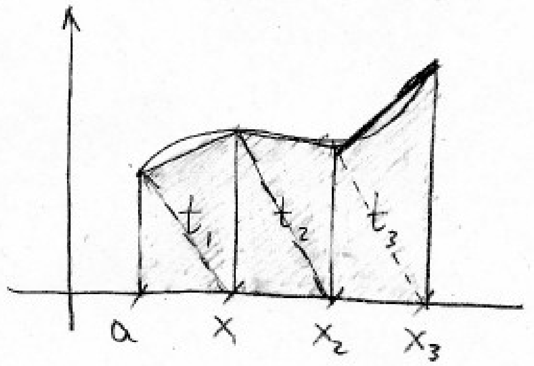


Illustration of T_3

we can geometrically see 6 triangles to get area by summing them.

$$t_1 = \tfrac{1}{2}f(x_0)\Delta x + \tfrac{1}{2}f(x_1)\Delta x$$
$$t_2 = \tfrac{1}{2}f(x_1)\Delta x + \tfrac{1}{2}f(x_2)\Delta x$$
$$t_3 = \tfrac{1}{2}f(x_2)\Delta x + \tfrac{1}{2}f(x_3)\Delta x$$

$$T_3 = t_1 + t_2 + t_3 = \tfrac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right) \quad //$$

Remark: It's not hard to see that $T_n = \frac{1}{2}(L_n + R_n)$

> **Def^n/** Let A_n be some approx. to $I = \int_a^b f(x)dx$ we define the absolute error in A_n to be
>
> $$I - A_n \equiv E_{A_n}$$

Some limitations and comparisons :

1. As n gets large the accuracy increases for each approx., modulo the limiting accuracy of the computer.
2. L_n and R_n have errors with opposite signs, and are halfed when we double n.
3. T_n & M_n are better approx. than L_n or R_n.
4. Errors in T_n and M_n are opposite in sign and are quarted when we double n
5. $E_{M_n} \approx \frac{-1}{2} E_{T_n}$

Besides these rules of thumb we can add some more concrete limiting comments,

> $Th^{\underline{m}}(3)$ Error Bounds. Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are errors in trap. and midpt. rule then
>
> $$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

I expect you to be able to apply these once given them, if there is some question on the test I would supply the the relevant inequality. It's upto you to use them,

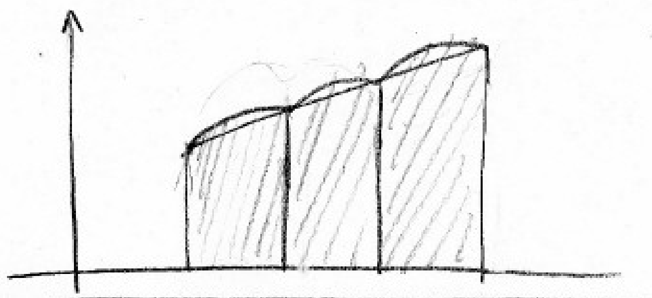
Example : Put an upper bound on E_{T_n} where T_n is used to approximate $\int_5^{11} e^x\,dx$. Notice $f(x) = e^x \Rightarrow f''(x) = e^x$ which is increasing since $f'''(x) > 0$ for $5 \leq x \leq 11$ thus $f''(5) \leq f''(x) \leq f''(11)$
$\Rightarrow f''(11)$ is an upper bound, specifically $f''(11) = e^{11} = K \geq |f''(x)|$

$$|E_T| \leq \frac{e^{11}(11-5)^3}{12n^2}$$

• Crucial to using the $Th^{\underline{m}}(3)$ is finding the K.

Simpson's Rule

Simpson's Rule approximates the area by integrating the area under a parabolic approx. of the function



S_3 for a linear function

Simpson's Rule
$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Where n must be even and $\Delta x = \frac{b-a}{n}$.

Proof/ See book.

Remark: Notice pattern $1, 4, 2, 4, 2, \ldots, 2, 4, 1$

How accurate is simpson's rule? Calculators base their numeric int. on it,

Th^m (4) ERROR BOUND FOR SIMPSON'S RULE: Let $|f''''(x)| \leq K$ for $a \leq x \leq b$ and let E_s be error in simpson's rule S_n

$$|E_{S_n}| \leq \frac{K(b-a)^5}{180n^4}$$

Remark: If we double n then the error is $\frac{1}{16}$ of what it was.

Example: If we approximate $\int_2^4 \sin\theta \, d\theta$ with S_n then find $|E_{S_n}| \leq$?
Notice that $f(\theta) = \sin(\theta) \Rightarrow f''''(\theta) = \sin(\theta)$ which has the property $|\sin(\theta)| \leq 1 \Rightarrow |f''''(\theta)| \leq 1$. Thus

$$|E_{S_n}| \leq \frac{1 \cdot (4-2)^5}{180n^4}$$

If we demand accuracy within $0.01 \Rightarrow |E_{S_n}| \approx 0.01$ thus

$$0.01 = \frac{32}{180n^4} \Rightarrow n = \left(\frac{32}{(0.01)180}\right)^{1/4} = 2.05 \Rightarrow \boxed{S_3 \text{ will certainly do the job.}}$$