Def^n/ A power series centered at zero has the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

Where $c_i \in \mathbb{R}\ \forall i$, these numbers are called the coefficients of the power series. power series centered at a has the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Remark $f(x)$ is a function with domain constructed from all x such that $f(x) \in \mathbb{R}$ (that is where the series converges)

Remark: A power series is a function which is defined pointwise by a series expansion.

E1 When $c_i = 1\ \forall i$ we have a geometric series with $a = 1$ and $r = x$

$$f(x) = \sum_{n=0}^{\infty} x^n = \boxed{1 + x + x^2 + \cdots = \frac{1}{1-x}}$$

this converges for each x in $(-1, 1)$

E2 Many times we can use substitution to rewrite a power series,

$$1 + x + 1 + x^2 + 2x + 1 + x^3 + 3x^2 + 3x + 1 + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

So this is again geometric series with $a = 1$ & $r = x+1$

$$1 + x + 1 + x^2 + 2x + 1 + \cdots = \frac{1}{1-(x+1)} = \frac{-1}{x} = f(x)$$

Which converges for $-1 < r = x+1 < 1 \iff -2 < x < 0 \leftarrow$ domain of f.

Th^m/(3) When $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series then $\exists$ 3 possibilities

i.) The series converges only when $x = a$

ii.) The series converges $\forall x \in \mathbb{R}$

iii.) $\exists R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

In (iii) R = the radius of convergence in this case we may have (for $a \geq 0$)

$$\text{dom}(f) = [a-R, a+R],\ (a-R, a+R),\ (a-R, a+R],\ [a-R, a+R)$$

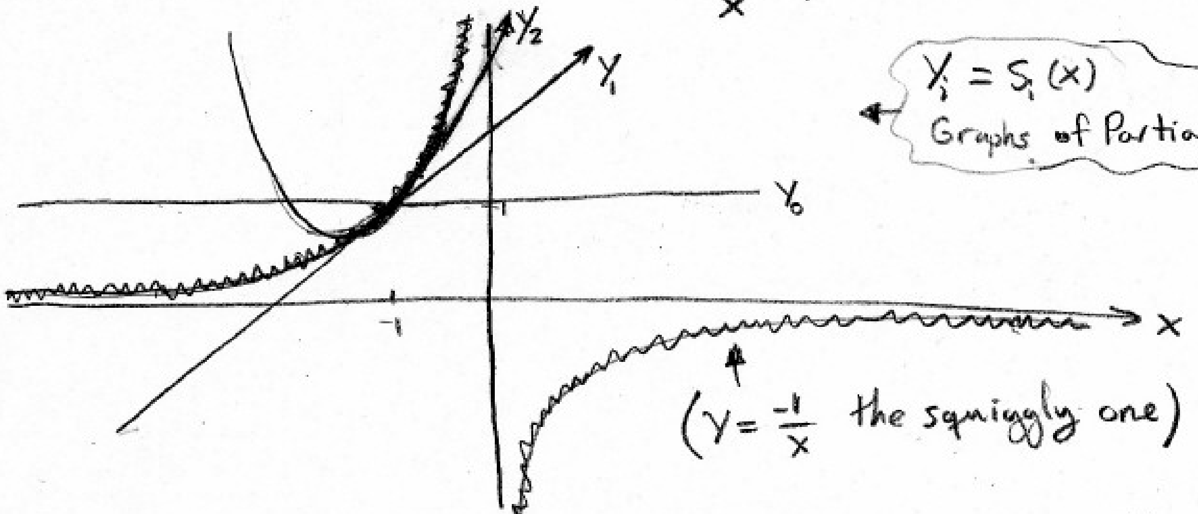
We call the dom(f) the interval of convergence. (I.o.C.)

Geometric Meaning of the Power Series

Let us focus on $1 + x + 1 + x^2 + 2x + 1 + \cdots = \sum_{n=0}^{\infty} (x+1)^n = \frac{-1}{x}$

this is a power series centered at $x = -1$ with I.O.C $(-2, 0)$ ← how do I know this? think about it

Question: How does the series relate to $\frac{-1}{x} = y$



$Y_i = S_i(x)$
Graphs of Partial Sums

$\left(y = \frac{-1}{x}\right.$ the squiggly one$\left.\right)$

Where $\frac{-1}{x} = \sum_{n=0}^{\infty} (x+1)^n = \lim_{n\to\infty} S_n(x)$, $S_n(x) = \sum_{i=1}^{n} (x+1)^i$

$$Y_0 = 1$$
$$Y_1 = 1 + (x+1) = x + 2$$
$$Y_2 = 1 + (x+1) + (x+1)^2 = x^2 + 3x + 3$$

- As n increases we take larger & larger polynomials to approximate $\frac{-1}{x}$ near $x = -1$, the higher n the closer to $\frac{-1}{x}$ we get.

Finding the Interval of Convergence

Step 1: Use the Ratio Test to find the interval for which the series converges (absolutely), usually $a - R < x < a + R$

Step 2: If $(a-R, a+R)$ is a finite interval then check the endpoints for convergence/divergence. This usually will entail a n^{th} term test or an alternating series test.

Step 3: Using the $Th^{m}(3)$ we're done. (If Step 2 has both endpt.'s divergent then I.O.C is $(a-R, a+R)$

Remark: this not a comprehensive advice, sometimes it'll be easier & sometimes you'll need to think or do some algebra before applying the steps.

$\boxed{E3}$ Find the I.O.C for $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}\,x^n}{3^n} = \sum_{n=0}^{\infty} a_n = S(x)$. STEP 1: RATIO TEST

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\,x^{n+1}}{3^{n+1}}\frac{3^n}{\sqrt{n}\,x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{3\sqrt{n}}x\right|$$

$$= \frac{1}{3}|x| \lim_{n\to\infty}\underbrace{\left|\frac{\sqrt{n+1}}{\sqrt{n}}\right|}_{\to 1} \quad \text{since, } \frac{\sqrt{n+1}}{\sqrt{n}} = \frac{n+1}{\sqrt{n^2+n}} = \frac{1+1/n}{\sqrt{1+1/n}} \to 1 \text{ as } n\to\infty.$$

Thus $L = \frac{1}{3}|x|$ so $L < 1$ if $\frac{1}{3}|x| < 1$ aka $|x| < 3$ and $L > 1$ if $|x| > 3$. Thus $S(x)$ converges on $(-3, 3)$.

STEP 2: Endpoints, test for conv./div at $x = \pm 3$

Consider thus

$$S(3) = \sum_{n=0}^{\infty}\frac{\sqrt{n}}{3^n}3^n = \sum_{n=0}^{\infty}\sqrt{n}$$: diverges since $\sqrt{n}\to\infty$ as $n\to\infty$ (using n^{th} term test)

$$S(-3) = \sum_{n=0}^{\infty}\frac{\sqrt{n}}{3^n}(-3)^n = \sum_{n=0}^{\infty}(-1)^n\sqrt{n}$$: diverges by n^{th} term test again $(-1)^n\sqrt{n} \not\to 0$ as $n\to\infty$.

Th^{m} (Term by term calculus on power series). If $S(x) = \sum C_n(x-a)^n$ has radius of conv. $R > 0$ then $S(x)$ is differentiable (or smooth) on $(a-R, a+R)$ and

a.) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} C_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}\left[C_n(x-a)^n\right] = \sum_{n=1}^{\infty} C_n n(x-a)^{n-1}$$

b.) $$\int\left[\sum C_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int C_n(x-a)^n dx = C + \sum_{n=0}^{\infty} C_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence for a. & b are also R although the I.O.C. may differ.

Remark: one must have a power series for this to work e.g. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n!x)}{n^2}$ converges while $\displaystyle\sum_{n=1}^{\infty}\frac{n!\cos(n!x)}{n^2}$ diverges

(This is an expansion in $\sin(\ )$ & $\cos(\ )$ not x^n it's not a power series.)