E4 Identify the function $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad -1 \le x \le 1$. Lets use the thⁿ to get geom. series,

$$f'(x) = 1 - x^2 + x^4 - \dots = \frac{a}{1-r} = \frac{1}{1+x^2} \quad \left(\text{where } \begin{matrix} a = 1 \\ r = -x^2 \end{matrix}\right)$$

Now we can integrate:

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} \Rightarrow f(x) = c + \tan^{-1}(x)$$

Now $f(0) = 0 \Rightarrow c + \tan^{-1}(0) = 0 \therefore \underline{c = 0}$, hence $\boxed{f(x) = \tan^{-1}(x)}$

> **Remark:** In the last example we were given a series and asked to identify what elementary fnct. it represented, it is more often the case we'll begin with some elementary fnct. and ask for it's power-series expansion. (a fnct that has a power series rep. is called "analytic")

E5 Find a power series rep. of $\ln(1+x) = f(x)$. Try diffⁿ

$$f'(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n \quad \left(\begin{matrix} \text{geom. series} \\ \text{with } a = 1 \\ \text{\& } r = -x \end{matrix}\right)$$

Now integrate term by term,

$$f(x) = c + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Find c by eval. f at $x = 0$, $f(0) = \ln(1) = \boxed{0 = c}$ thus

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \boxed{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots = \ln(1+x)}$$

E6 Find power series expansion of $f(x) = \frac{1}{(1+x)^2}$

$$\int \frac{1}{(1+x)^2}\,dx = \frac{-1}{1+x} + c \quad , \text{ note } f(0) = 1 \Rightarrow c = 2.$$

$$\int f(x)\,dx = 2 - \frac{1}{1+x} = 2 - \left(\sum_{n=0}^{\infty} (-x)^n\right) \quad : \text{geom. series } \begin{matrix} a = 1 \\ r = -x \end{matrix}$$

$$f(x) = \frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\left[2 - \sum_{n=0}^{\infty} (-x)^n\right] = \sum_{n=1}^{\infty} n(-x)^{n-1} = \boxed{1 - 2x + 3x^2 + \dots = \frac{1}{(1+x)^2}}$$

E7 Power series can also help with some formidable integrals!

$$\int \frac{\tan^{-1}(x)}{x}\,dx = f(x)$$

$$f'(x) = \frac{\tan^{-1}(x)}{x} = \frac{1}{x}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \cdots$$

$$\int f'(x)\,dx = f(x) = C + x - \frac{x^3}{9} + \frac{x^5}{25} - \cdots$$

Thus $\boxed{\int \frac{\tan^{-1}(x)}{x}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{2n+1}}{(2n+1)^2}}$

Remark: We could approximate definite integrals via their power series expansions, see example 8 in text if you're interested. We'll do more of these once we've mastered the Taylor Series, see 231 - 232.

E8 Find power series expansion of $f(x) = \ln(1+x^2)$. Notice,

$$f'(x) = \left(\frac{1}{1+x^2}\right)2x$$

Now we can see how to apply geometric series result, $a = 2x$, $r = -x^2$

$$f'(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (2x)(-x^2)^n = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$$

Now recover $f(x)$ by integrating, $f(x) = \int f'(x)\,dx$

$$\begin{aligned} f(x) &= \int \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}\,dx \\ &= \sum_{n=0}^{\infty} (-1)^n 2 \int x^{2n+1}\,dx \\ &= \sum_{n=0}^{\infty} (-1)^n 2\left(\frac{x^{2n+2}}{2n+2}\right) + C \end{aligned}$$

note $f(0) = 0 \Rightarrow C = 0$

Therefore $\boxed{f(x) = \frac{2x}{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2} = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \cdots}$