Integrals involving powers of $\sin(\theta)$ and $\cos(\theta)$

I.) Odd powers can be calculated via the standard trick below,

$$
\begin{aligned}
\int \sin^3\theta\, d\theta &= \int \sin^2\theta \sin\theta\, d\theta \\
&= \int (1-\cos^2\theta)\sin\theta\, d\theta \qquad \begin{cases} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{cases} \\
&= \int (1-u^2)(-du) \\
&= -u + \tfrac{1}{3}u^3 + C \\
&= \boxed{-\cos\theta + \tfrac{1}{3}\cos^3\theta + C = \int \sin^3\theta\, d\theta}
\end{aligned}
$$

Almost the same calculation works for $\cos^3\theta$

$$
\begin{aligned}
\int \cos^3\theta\, d\theta &= \int \cos^2\theta \cos\theta\, d\theta \\
&= \int (1-\sin^2\theta)\cos\theta\, d\theta \qquad \begin{cases} u = \sin\theta \\ du = \cos\theta\, d\theta \end{cases} \\
&= \int (1-u^2)\, du \\
&= u - \tfrac{1}{3}u^3 + C \\
&= \boxed{\sin\theta - \tfrac{1}{3}\sin^3\theta + C = \int \cos^3\theta\, d\theta}
\end{aligned}
$$

The strategy can also be used to calculate larger odd powers of $\sin\theta$ or $\cos\theta$. Check out how $\cos^5\theta$ can be dealt with,

$$
\begin{aligned}
\int \cos^5\theta\, d\theta &= \int (\cos^2\theta)^2 \cos\theta\, d\theta \\
&= \int (1-\sin^2\theta)^2 \cos\theta\, d\theta \quad ; \text{ let } u = \sin\theta,\ du = \cos\theta\, d\theta \\
&= \int (1-u^2)^2\, du \\
&= \int (1-2u^2+u^4)\, du \\
&= u - \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5 + C \\
&= \boxed{\sin\theta - \tfrac{2}{3}\sin^3\theta + \tfrac{1}{5}\sin^5\theta + C = \int \cos^5\theta\, d\theta}
\end{aligned}
$$

You might try $\int \sin^5\theta\, d\theta$ as an exercise. Next I'll show you how to calculate the integral of even powers of $\sin\theta$ or $\cos\theta$.

II.) We'll need the trig. identities $\sin^2(x) = \frac{1}{2}(1-\cos 2x)$ and $\cos^2(x) = \frac{1}{2}(1+\cos 2x)$

$$\int \sin^2\theta \, d\theta = \int \tfrac{1}{2}(1-\cos 2\theta)\, d\theta \quad : \text{ let } u = 2\theta \text{ then } d\theta = \tfrac{1}{2}du$$
$$= \int \tfrac{1}{2}(1-\cos u)\frac{du}{2}$$
$$= \int \left(\tfrac{1}{4} - \tfrac{1}{4}\cos u\right) du$$
$$= \tfrac{1}{4}u - \tfrac{1}{4}\sin u + C$$
$$= \boxed{\tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta) + C = \int \sin^2\theta \, d\theta}$$

Very similarly:

$$\int \cos^2\theta \, d\theta = \int \tfrac{1}{2}(1+\cos 2\theta)\, d\theta \quad : \text{ let } u = 2\theta \text{ so } d\theta = \tfrac{1}{2}du$$
$$= \int \tfrac{1}{2}(1+\cos u)\tfrac{1}{2}du$$
$$= \tfrac{1}{4}\int (1+\cos u)\, du$$
$$= \tfrac{1}{4}(u + \sin u) + C$$
$$= \boxed{\tfrac{1}{2}\theta + \tfrac{1}{4}\sin(2\theta) + C = \int \cos^2\theta \, d\theta}$$

We can extend these techniques to higher even powers,

$$\int \cos^4\theta \, d\theta = \int \left[\tfrac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$$
$$= \tfrac{1}{4}\int \left(1 + 2\cos 2\theta + \cos^2(2\theta)\right) d\theta \qquad \text{let } u = 2\theta, \ d\theta = \tfrac{1}{2}du$$
$$= \tfrac{1}{4}\int \left(1 + 2\cos u + \cos^2(u)\right)\frac{du}{2}$$
$$= \tfrac{1}{8}\left(u + 2\sin u + \tfrac{1}{2}u + \tfrac{1}{4}\sin(2u)\right) + C$$

using the result above with u in the place of θ.

$$= \boxed{\tfrac{1}{8}\left(\tfrac{3}{2}(2\theta) + 2\sin(2\theta) + \tfrac{1}{4}\sin(4\theta)\right) + C}$$

This time I used the integral of $\cos^2\theta$ we just did.