Sequences, an Introduction:

Def^{n}/ A sequence is a list of numbers $\{a_1, a_2, a_3, \ldots\} \equiv \{a_n\}_{n=1}^{\infty} \equiv \{a_n\}$.
For each $n \in \mathbb{N}$ we assign $a_n \in \mathbb{R}$ (it's a fnct. from $\mathbb{N}$ to $\mathbb{R}$).
Usually we take sequences starting from $n = 1$ however we also consider

$$\{b_0, b_1, b_2, b_3, \ldots\} = \{b_n\}_{n=0}^{\infty}$$

$$\{c_3, c_4, c_5, \ldots\} = \{c_n\}_{n=3}^{\infty}$$

the above to be sequences.

E1 $\{a_n\}_{n=0}^{\infty} = \{1, -\sqrt{2}, \sqrt{3}, -\sqrt{4}, \ldots\} = \{a_0, a_1, a_2, a_3, \ldots\}$
What's a formula for a_k? well inspection reveals:

$$\boxed{a_k = \sqrt{k+1}}$$

E2 $\{b_n\}_{n=1}^{\infty} = \{2, 2.5, 3.\overline{33}, 4.25, \ldots\}$ then we can see the formula for an arbitrairy term in this sequence is $\boxed{b_k = k + \frac{1}{k}}$

E3 $\{c_n\}_{n=1}^{\infty} = \{0, \frac{-1}{2}, \frac{2}{3}, \frac{-3}{4}, \ldots\}$ what is c_n? Well, the sequence is alternating ... the formula is $\boxed{c_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)}$

Remark: not all sequences have simple formulas as above, some are defined recursively or even randomly any way I should mention Fibonacci

$$\{1, 1, 2, 3, 5, 8, 13, \ldots\} \quad a_1 = a_2 = 1 \quad \text{and} \quad a_k = a_{k-1} + a_{k-2}$$

Generally we'll focus on sequences like **E1**, **E2** & **E3**.

New sequences from old: b.c. sequences are functions we already know how to add, substract, multiply, divide we do it point-wise. Let $\{a_n\}, \{b_n\}$ be sequences

i.) $\{a_n\} = \{b_n\}$ iff $a_n = b_n$ for each n.

ii.) $\{(a+b)_n\} = \{a_n + b_n\}$

iii) $\{(ab)_n\} = \{a_n \cdot b_n\}$

iv.) $\{(a/b)_n\} = \{a_n / b_n\}$ (provided $b_n \neq 0$ for all n)

v.) $\{(c \cdot a)_n\} = \{c \cdot a_n\}$

Def^n/ A sequence $\{a_n\}$ has limit L which is denoted

$$\lim_{n\to\infty} a_n = L \quad \text{OR} \quad a_n \longrightarrow L \text{ as } n\to\infty$$

If we can make a_n arbitrarily close to L for sufficiently large n. When the limit exists we say $\{a_n\}$ converges to L otherwise we say $\{a_n\}$ diverges (or is divergent)

Remark: this is the same as the limit of a fnct which leads to

Th^m If $f(n) = a_n$ for each $n \in \mathbb{Z}$, $n \geq n_0$ then

$$\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L$$

Additionally if the limit of f diverges then the limit of $\{a_n\}$ diverges in the same way.

Properties of Limits:

Let $a_n \longrightarrow A$ and $b_n \longrightarrow B$ as $n \to \infty$ then

i) $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

ii) $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

iii) $\lim_{n\to\infty} (c \cdot a_n) = c \cdot A$ $\quad (c \in \mathbb{R})$

iv.) $\lim_{n\to\infty} ({}^{a_n}/_{b_n}) = {}^{A}/_{B}$ $\quad (B \neq 0)$

v.) $\lim_{n\to\infty} (c) = c$ $\quad$ (duh.)

E4 Find limit of $a_n = \dfrac{n+3}{n^2+5n+6}$ (could factor, ÷ by n^2, or L'Hopital's Rule)

$$\lim_{n\to\infty}\left(\frac{n+3}{n^2+5n+6}\right) = \lim_{n\to\infty}\left(\frac{1/n + 3/n^2}{1 + 5/n + 6/n^2}\right) = \frac{0}{1} = 0.$$

Limits of Functions	$\Rightarrow$	Limits of Sequences
$\lim\limits_{x\to\infty}(x^p) = \begin{cases} \infty & p>0 \\ 1 & p=0 \\ 0 & p<0 \end{cases}$	$\Rightarrow$	$\lim\limits_{n\to\infty}(n^p) = \begin{cases} \infty & p>0 \\ 1 & p=0 \\ 0 & p<0 \end{cases}$
$\lim\limits_{x\to\infty}(\ln(x)) = \infty$	$\Rightarrow$	$\lim\limits_{n\to\infty}(\ln(n)) = \infty$
$\lim\limits_{x\to\infty}(e^x) = \infty$	$\Rightarrow$	$\lim\limits_{n\to\infty}(e^n) = \infty$
$\lim\limits_{x\to\infty}(\tan^{-1}(x)) = \frac{\pi}{2}$	$\Rightarrow$	$\lim\limits_{n\to\infty}(\tan^{-1}(n)) = \frac{\pi}{2}$.

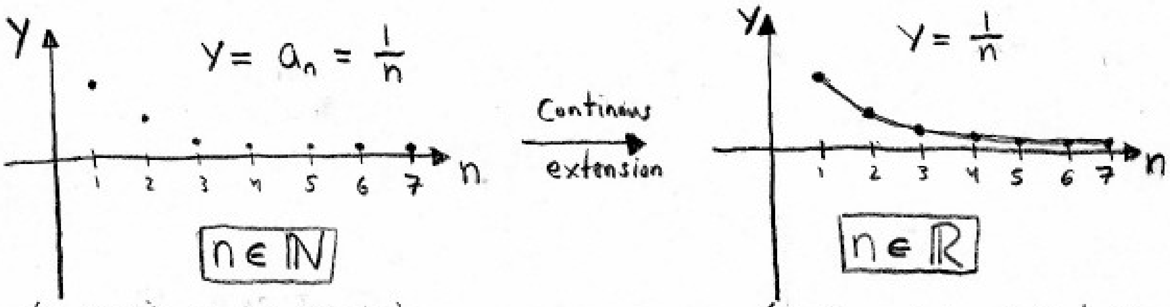
- In each case we transfer what we know about the continuous case (x) to obtain the same result for the discrete case (n). Technically this is crucial because I can apply L'Hopital's to functions of x but I ought not do it to functions of the discrete variable n (how can you do limits $h\to 0$ inside $\mathbb{N}$?) fortunately this difficulty is easily avoided if we just extend n to be continous. It's ok to do this precisely because of the Th^{m} on 197. You can apply L'Hopital's rule to functions of n in my course, but it would be good to write in the margin "extending n to be a continous variable"

$\boxed{E5}$

$$\lim_{n\to\infty}(ne^{-n}) = \lim_{n\to\infty}\left(\frac{n}{e^n}\right) \underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{n\to\infty}\left(\frac{1}{e^n}\right) = \boxed{0}.$$

(Extending n to be a continous variable.)

- Graphical Meaning of "Extending to continous variable" for $\frac{1}{n}$,



$\boxed{n\in\mathbb{N}}$ (No limits $\Rightarrow$ No Calculus) (Finite limits)

$\boxed{n\in\mathbb{R}}$ (limits, calculus, L'Hopital's...)

Th^{m}/ Squeeze Th^{m} for Sequences : If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(c_n) = L$ then

$$\lim_{n\to\infty}(b_n) = L$$

- especially useful for limits involving sine or cosine.

E6 Find $\lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right)$.

Notice that: $-1 \le \sin(n) \le 1$

And for $n \ge 1$, $\frac{-1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$

Then clearly $\pm\frac{1}{n} \longrightarrow 0$ as $n \to \infty$. Thus By Squeeze Th^{m} so must $\frac{\sin(n)}{n} \longrightarrow 0$ as $n \to \infty$.

$$\lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right) = 0$$

Th^{m} (Absolute Convergence for Sequences)

If $\lim_{n\to\infty}|a_n| = 0$ then $\lim_{n\to\infty}(a_n) = 0$

E7 Calculate $\lim_{n\to\infty}\frac{(-1)^n}{n}$ if possible.

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 \Rightarrow \boxed{\lim_{n\to\infty}\left(\frac{(-1)^n}{n}\right) = 0}$$

E8 • Important Example (not connected with Th^{m}'s above)

$$\lim_{n\to\infty}(r^n) = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & -1 < r < 1 \\ \text{d.n.e} & r \le -1 \end{cases}$$

We'll use this later on the "geometric series"

Def^n/ A sequence $\{a_n\}_{n_0}^{\infty}$ is increasing if $a_n < a_{n+1}$ for all $n \geq n_0$
A sequence $\{b_n\}_{n_0}^{\infty}$ is decreasing if $b_n > b_{n+1}$ for all $n \geq n_0$
A sequence is monotonic if its either inc. or dec.

Remark: If $f(n) = a_n$ and $f(x)$ is increasing on $x \geq 1$ then its clear that $\{a_n\}_{n=1}^{\infty}$ is an inc. sequence.

Def^n/ A sequence $\{a_n\}_{n=n_0}^{\infty}$ is bounded above if $\exists M \in \mathbb{R}$ such that $a_n \leq M$ for $n \geq n_0$. Likewise $\{a_n\}_{n=n_0}^{\infty}$ is bounded below if $\exists m \in \mathbb{R}$ such that $a_n \geq m$ for $n \geq n_0$. When $\{a_n\}_{n=n_0}^{\infty}$ is bounded above AND below it said to be bounded.

E9 $a_n = 3 + \frac{1}{n}$ is bounded. Why? Well notice that

$$3 < 3 + \frac{1}{n} \leq 4 \quad \text{for } n \geq 1.$$

Notice that $a_n \longrightarrow 3$ as $n \longrightarrow \infty$. This is no accident,

Th^m/ (Monotonic Sequence Th^m)
Every bounded monotonic sequence is convergent

E10 $n! = n(n-1)(n-2)\cdots(4)(3)(2)(1)$ "n factorial"

How does the sequence $a_n = \frac{1}{n!}$ behave? Well notice that $0 < \frac{1}{n!} \leq 1$ for $n \geq 1$ so its bounded,

$$\frac{1}{(n+1)!} = \frac{1}{(n+1)n(n-1)\cdots 3\cdot 2\cdot 1} < \frac{1}{n(n-1)\cdots 3\cdot 2\cdot 1} = \frac{1}{n!} \quad (\text{for } n > 1)$$

We see $\frac{1}{n!} > \frac{1}{(n+1)!}$ the sequence $\{\frac{1}{n!}\}$ is decreasing. Thus $\{\frac{1}{n!}\}_{n=1}^{\infty}$ is a bounded monotonic sequence, it converges. In fact

$$\boxed{\lim_{n \to \infty} \left(\frac{1}{n!}\right) = 0} \longleftarrow$$ haven't argued why but its not hard to believe.