Series (§ 8.2)

The primary question we wish to answer is when does the sum of a sequence add up to a real number?

$$a_1 + a_2 + \cdots + a_n + \cdots = S$$

The sum of a sequence is called a series. We need to make sense of this more carefully.

Def^n/ The n^{th} partial sum of $\{a_n\}$ is $S_n = \sum_{i=1}^{n} a_i$

Def^n/ The series (relative to $\{a_n\}$) is the limit of the partial sums

$$S = \lim_{n\to\infty} S_n = a_1 + a_2 + a_3 + \cdots$$

When this limit exists we say the series converges. Otherwise we say the series S diverges.

$\boxed{E1}$ $\{a_n\} = \{\frac{1}{n}\}$ has a divergent series $S = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ this is called the harmonic series.

$\boxed{E2}$ $\{a_n\} = \{1\}$ gives div. series, $S = 1 + 1 + 1 + \cdots$

$\boxed{E3}$ $\{\frac{1}{n^2}\} = \{a_n\}$ is convergent $S = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$

We'll Explain why later.

Geometric Series:

$$a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } -1<r<1$$

Proof: Strategy find S_n explicitly then let $n \to \infty$ to find series,

$$\begin{aligned} S_n &= a + \cancel{ar} + \cdots + ar^{n-1} \\ -\,rS_n &= \cancel{ar} + \cancel{ar^2} + \cdots + ar^n \\ \hline S_n - rS_n &= a - ar^n \implies S_n = \frac{a(1-r^n)}{1-r} \end{aligned}$$

$$S = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = a \cdot \lim_{n\to\infty}\left(\frac{1}{1-r}\right) - \frac{a}{1-r}\underbrace{\lim_{n\to\infty}(r^n)}_{0\ (-1<r<1)}$$

$$\therefore \boxed{S = \frac{a}{1-r}} \quad \text{for } |r| < 1 \leftarrow \text{(interval of convergence)}$$

Remark: The geometric series is everywhere, you'll see.

E4 Find the exact fraction that gives $2.161616 = 2.\overline{16}$. We can use the geometric series:

$$2.\overline{16} = 2 + \underbrace{\frac{16}{100} + \frac{16}{(100)^2} + \frac{16}{(100)^3} + \cdots}_{} = 2 + \frac{16/100}{1-\frac{1}{100}}$$

$$a = \frac{16}{100} \text{ and } r = \frac{1}{100}$$

$$\frac{16}{100\left(1-\frac{1}{100}\right)} = \frac{16}{100-1} = \frac{16}{99} = 0.\overline{16} \qquad \therefore \boxed{2.\overline{16} = 2 + \frac{16}{99}}$$

E5 Calculate $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ for $|x| < 1$.

Let $a = 1$ and $r = x$ this is geometric series with that identification,

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

Telescoping Series:

This is another class of series we can calculate explicitly, no general formula like the geometric series, here we must do algebra,

E6 Calculate $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = S$

$$S_n = \left(1 - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}}\right) + \cdots + \left(\cancel{\frac{1}{\sqrt{n-1}}} - \cancel{\frac{1}{\sqrt{n}}}\right) + \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}$$

$$S = \lim_{n\to\infty}\left(1 - \underbrace{\frac{1}{\sqrt{n+1}}}_{0}\right) = \boxed{1 = S}$$

E7 $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = S$ <u>use partial-fractions</u> $\frac{4}{(4n-3)(4n+1)} = \frac{1}{4n-3} - \frac{1}{4n+1}$

$$S_n = \left(\frac{1}{4-3} - \frac{1}{4+1}\right) + \left(\frac{1}{8-3} - \frac{1}{8+1}\right) + \left(\frac{1}{12-3} - \frac{1}{12+1}\right) + \cdots + \left(\frac{1}{4(n-1)-3} - \frac{1}{4(n-1)+1}\right) + \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$$

$$= \left(1 - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{9}} - \frac{1}{13}\right) + \cdots + \left(\cancel{\frac{1}{4n-7}} - \cancel{\frac{1}{4n-3}}\right) + \left(\cancel{\frac{1}{4n-3}} - \frac{1}{4n+1}\right)$$

$$= 1 - \frac{1}{4n+1}$$

$$\therefore S = \lim_{n\to\infty}\left(1 - \underbrace{\frac{1}{4n+1}}_{0}\right) = \boxed{1 = S}$$

E8 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ See Example 6 pg. 571

$\boxed{\text{Th}^{\underline{m}}(6) \text{ If } \sum_{n=1}^{\infty} a_n \text{ is convergent then } \lim_{n\to\infty} a_n = 0}$

Th$^{\underline{m}}$(7) ($n^{\underline{th}}$ term divergence test)
if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent. ← important!

Remark: this th$^{\underline{m}}$ proves that $1+1+1+\cdots$ does not converge since $a_n = 1$ and $\lim_{n\to\infty} a_n = 1 \neq 0 \therefore \sum_{n=1}^{\infty} a_n$ diverges.

Remark: Th$^{\underline{m}}$(7) is the $1^{\underline{st}}$ and easiest of the divergence tests. We'll find more sophisticated tests in coming lectures, notice that

$$\lim_{n\to\infty} a_n = 0 \quad \not\Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges}$$

(If it did this chapter would be alot easier & $\sum_{n=1}^{\infty} \frac{1}{n}$ wouldn't diverge!)

Th$^{\underline{m}}$(8) Given that $\sum^{\infty} a_n$ and $\sum^{\infty} b_n$ are convergent and $c \in \mathbb{R}$ then all of the following are also convergent series.

i) $\sum_{n=1}^{\infty} c a_n = c \cdot \sum_{n=1}^{\infty} a_n$

ii.) $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

$\boxed{E9}$ $$\sum_{n=1}^{\infty}\left(ar^{n-1} + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) \overset{(*)}{=} \sum_{n=1}^{\infty} ar^{n-1} + \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$$

$$= \frac{a}{1-r} + 1$$

Geom. Series   See $\boxed{E6}$

bc these are convergent the (*) equality is an ok step.

$\boxed{E10}$ $$\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{\text{telescopes to } \underline{1} \text{ in fact.}} \neq \cancel{\sum_{n=1}^{\infty} \frac{1}{n}}^{\infty} - \cancel{\sum_{n=1}^{\infty} \frac{1}{n+1}}^{\infty}$$

these are divergent thus Th$^{\underline{m}}$(8) cannot be used.

(Th$^{\underline{m}}$(8)'s alterego)