Trigonometric Substitutions (§ 5.7 the book is weak on this)

Sometimes a substitution involving a trigonometric function is useful. Mainly because trig. functs. have nice identities:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

Or dividing by $\cos^2\theta$ we obtain

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

These identities together with differential identities $d(\sin\theta) = \cos\theta\, d\theta$ and $d(\tan\theta) = \sec^2\theta\, d\theta$ exc... will prove to be especially useful to remove unwanted radicals from a given integral. By "remove" we really mean it changes the problem so that we can solve it.

E1

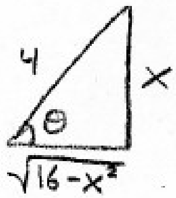
$$\int\sqrt{16-x^2}\,dx = \int(4\cos\theta)(4\cos\theta\,d\theta) \leftarrow \boxed{\begin{aligned} x &= 4\sin\theta \\ dx &= 4\cos\theta\,d\theta \\ \sqrt{16-x^2} &= \sqrt{16(1-\sin^2\theta)} = 4\cos\theta \end{aligned}}$$

$$= 16\int\cos^2\theta\,d\theta$$

$$= \frac{16}{2}\int(1+\cos(2\theta))\,d\theta$$

$$= 8\left(\theta + \tfrac{1}{2}\sin(2\theta)\right) + C$$

$$= \boxed{8\sin^{-1}\left(\frac{x}{4}\right) + 4\sin\left(\sin^{-1}\left(\frac{x}{4}\right)\right) + C}$$



E2

$$\int\frac{\sqrt{9-x^2}}{x^2}\,dx = 9\int\frac{\cos\theta\cos\theta\,d\theta}{9\sin^2\theta} \leftarrow \boxed{\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta\,d\theta \\ \sqrt{9-x^2} &= \sqrt{9-9\sin^2\theta} = 3\cos\theta \end{aligned}}$$

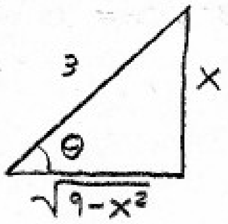
$$= \int\cot^2\theta\,d\theta$$

$$= \int(\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$

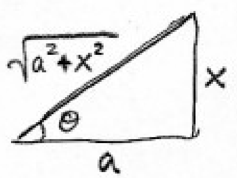
$$= \frac{-3\cos\theta}{3\sin\theta} - \theta + C$$

$$= \boxed{\frac{-\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$



TRIG SUBSTITUTION (Taken from § 7.4 TAYLOR)

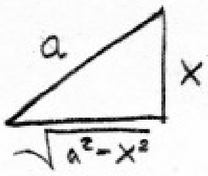
We reduce a^2+x^2, a^2-x^2 and x^2-a^2 to a single term $\Rightarrow$ square roots simplify. Often one of the subst. below is helpful



$$x = a\tan\theta$$

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = a\sqrt{\sec^2\theta} = a|\sec\theta|$$

$$a^2+x^2 = a^2\sec^2\theta$$

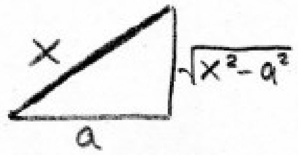


$$x = a\sin\theta$$

$$\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2\theta)} = a\sqrt{\cos^2\theta} = a|\cos\theta|$$

$$a^2-x^2 = \cos^2\theta$$

{ $x = a\cos\theta$ also works }



$$x = a\sec\theta$$

$$\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta-1)} = a\sqrt{\tan^2\theta} = a|\tan\theta|$$

$$x^2-a^2 = a^2\tan^2\theta$$

E3 Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$. We'll try the 1st subst.

$x = 2\tan\theta \implies 4+x^2 = 4+4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$:

$dx = 2\sec^2\theta\, d\theta$ & $4+x^2 = 4\sec^2\theta$ hence,

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \sec(\theta)\, d\theta$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C}$$

let $u = \sec\theta + \tan\theta$

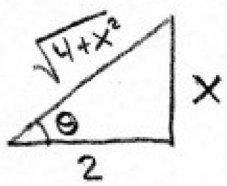
$du = (\sec\theta\tan\theta + \sec^2\theta)d\theta$

$du = \sec\theta(\sec\theta + \tan\theta)d\theta$

$\frac{du}{u} = \sec\theta\, d\theta$

Using figure below,

this Δ corresponds to the subst. $x = 2\tan\theta$

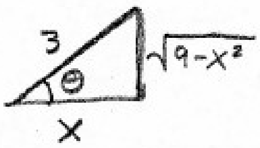


$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj.}} = \frac{\sqrt{4+x^2}}{2}$$

$$\tan\theta = \frac{\text{OPP}}{\text{ADJ.}} = \frac{x}{2}$$

E4 $\displaystyle\int \frac{x^3\,dx}{\sqrt{9-x^2}}$ for $-3 < x < 3$.

Could use $x = 3\sin\theta$. try it for yourself.



$x = 3\cos\theta$ & $dx = -3\sin\theta\, d\theta$

$9 - x^2 = 9 - 9\cos^2\theta = 9\sin^2\theta$

$$\int \frac{x^3\,dx}{\sqrt{9-x^2}} = \int \frac{(3\cos\theta)^3(-3\sin\theta\, d\theta)}{\sqrt{9\sin^2\theta}}$$

since $\dfrac{\sin\theta}{|\sin\theta|} = 1$ for $0 < \theta < 90°$

$$= -\int 27\cos^3\theta\, d\theta$$

$$= -27\int (1-\sin^2\theta)\cos\theta\, d\theta$$

$$= 27\int (u^2 - 1)\, du \qquad \leftarrow \boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\, d\theta \end{array}}$$

$$= 27\left[\frac{u^3}{3} - u\right] + c$$

$$= 9\sin^3\theta - 27\sin\theta + c$$

note $\sin\theta = \dfrac{\sqrt{9-x^2}}{3}$

$$= 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - 27\left(\frac{\sqrt{9-x^2}}{3}\right) + c$$

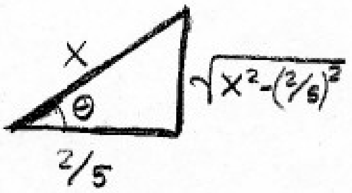
$$= \boxed{-9\sqrt{9-x^2} + \frac{1}{3}(9-x^2)^{3/2} + c}$$

Subtle Remark: I used $0 < \theta < 90° \Rightarrow |\sin\theta| = \sin\theta > 0$ implicitly when I simplified $\sqrt{\sin^2\theta} = \sin\theta$. In principal you might find $\sqrt{\sin^2\theta} = -\sin\theta$, but that is not the case here, because $\sin\theta > 0$ and squareroots are by convention positive. Notice that $0 < \theta < 90°$ follows from $-3 < x = 3\sin\theta < 3 \Rightarrow -1 < \sin\theta < 1$ $\Rightarrow 0 < \theta < 90°$. From the beginning I knew that $|x| < 3$ because otherwise the integrand is not real-valued.

E5 $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}}$ for $x > 2/5$

look's like $\sqrt{x^2-a^2}$ but not quite yet, do algebra on it.

$$\sqrt{25x^2-4} = \sqrt{25(x^2-4/25)}$$
$$= 5\sqrt{x^2-(2/5)^2}$$ suggests we use $x = \frac{2}{5}\sec\theta$



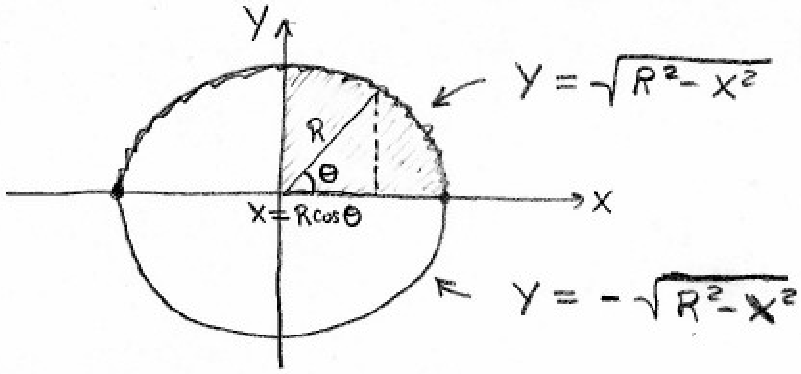
$$x = \frac{2}{5}\sec(\theta)$$
$$dx = \frac{2}{5}\sec(\theta)\tan\theta \, d\theta =$$
$$x^2-(2/5)^2 = (2/5)^2[\sec^2\theta - 1] = \left(\frac{2}{5}\tan\theta\right)^2$$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \frac{1}{5}\int \frac{dx}{\sqrt{x^2-2/5}}$$
$$= \frac{1}{5}\int \frac{\frac{2}{5}\sec(\theta)\tan(\theta)\,d\theta}{\sqrt{(\frac{2}{5}\tan\theta)^2}}$$
$$= \frac{1}{5}\int \sec\theta \, d\theta$$
$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$ using **E3**
$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{x^2-(2/5)^2}}{2/5}\right| + C$$
$$= \boxed{\frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C}$$

Remark: Trig. subst. requires some work for most people. Unfortunately there are not enough practice problems in your text so I'll supply some extra recommended hwk. on this topic.

AREA OF CIRCLE

A circle of radius R is given by $x^2 + y^2 = R^2$.



By symmetry if we find are of upper quadrant we can just multiply by 4 to get the area of the whole circle.

$$\int_0^R \sqrt{R^2 - x^2}\, dx = \int_0^{\pi/2} R\cos\theta \cdot R\cos\theta\, d\theta$$

Use trig. substitution
$x = R\sin\theta$
$dx = R\cos\theta\, d\theta$
Under this subst. we have,
$\sqrt{R^2 - x^2} = R\cos\theta$

$$= R^2 \int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= R^2 \int_0^{\pi/2} \frac{1}{2}(1 + \cos(2\theta))\, d\theta$$

$$= \frac{R^2}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Big|_0^{\pi/2}$$

$$= \frac{R^2}{2}\left[\frac{\pi}{2} + \frac{1}{2}\underbrace{\sin(\pi)}_{0} - \frac{1}{2}\underbrace{\sin(0)}_{0}\right]$$

$$= \frac{\pi R^2}{4}$$

Multiply by 4 and we find the area of circle is $\boxed{\pi R^2}$

A brief look ahead:

- There is a connection between the subst. used here and polar coord.

$$x = r\cos\theta \qquad y = r\sin\theta$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$

polar coordinates more naturally describe the circle. In calculus III you'll learn that the area of a circle comes from

$$\int_0^{2\pi}\int_0^R r\,dr\,d\theta = \left(\int_0^R r\,dr\right)\left(\int_0^{2\pi} d\theta\right) = \left(\frac{1}{2}R^2\right)(2\pi) = \pi R^2$$

Why did I integrate $r\,dr\,d\theta$? See calc. III for the answer.