Extra Credit Project

- Show work, even if I did it in notes or elsewhere

Level One (3pts.)

1.) $\int \left(\frac{2}{\sqrt{x}} + 3\right) dx$

2.) $\int \exp(kx)\, dx$

3.) $\int_0^{\pi/2} \sec\theta \tan\theta\, d\theta$

4.) $\int u^2(1+au^3)^7\, du$

5.) $\int \cos\theta\, e^{\sin\theta}\, d\theta$

6.) $\int \sec^2\theta\, e^{\tan\theta}\, d\theta$

7.) $\int \frac{2t}{t^2-25}\, dt$

8.) $\int \frac{1}{x \ln(x)}\, dx$

9.) $\int \frac{1}{\theta} \cos(\ln\theta - A)\, d\theta$

10.) $\int z^{\tan(x)} \sec^2(x)\, dx$

11.) $\int x\, 3^{x^2}\, dx$

12.) $\int \frac{1}{v} \tan(\ln(v))\, dv$

13.) $\int \frac{1}{r} \csc^2(1+\ln(r))\, dr$

14.) $\int_{-4}^{-3} \left(\frac{1}{x}\right) dx$

15.) $\frac{d}{dx} \int_{\sin(x)}^{x} g(u)\, du$

Level Two (3pts.)

16.) $\int \frac{x}{6-3x}\, dx$

17.) $\int x e^{3x}\, dx$

18.) $\int \tan^2\theta\, d\theta$

19.) $\int \cos(\pi t + k)\, dt$

20.) $\int \frac{1}{x^2-1}\, dx$

21.) $\int \frac{1}{x^2+5x+6}\, dx$

22.) $\int \frac{1}{\theta} \log_A(\theta)\, d\theta$

23.) $\int \frac{3}{4+3x^2}\, dx$

Level Three (2pts.)

24.) $\int e^{\theta} \cos(\theta)\, d\theta$

25.) $\int \cos^4(\theta)\, d\theta$

26.) $\int \frac{1}{\sqrt{9-x^2}}\, dx$

27.) $\int \frac{1}{1+x^2} \left(\tan^{-1}(x) + 1\right) dx$

28.) $\int_0^{\infty} x e^{-x}$

29.) $\int_0^{\infty} x^2 e^{-x}\, dx$

30.) $\int_0^{\infty} x^n e^{-x}\, dx \quad (n \in \mathbb{N})$

Level Four (2pts)

31.) $\int \sin^k\theta\, d\theta$ for $k = 1, 2, 3, 4, 5, 6$.

32.) $\int \sec^k\theta\, d\theta$ for $k = 1, 2, 3, 4$

33.) $\int \frac{x^2}{(x-1)(x-2)}\, dx$

34.) $\int \sqrt{a^2+x^2}\, dx$

35.) $\int \sin(x) \cos(3x)\, dx$

36.) $\int (\cos^4\theta - \sin^4\theta)\, d\theta$

Level Five (2pts.)

37.) $\int_{-1}^{1} \frac{\sinh(x)}{\cos(x)}\, dx$

38.) $\int \frac{1}{\cos^4(x) \sin^6(x)}\, dx$ (use $t = \tan(x)$ subst.)

39.) $\int \frac{1}{4\sin(x) + 3\cos(x) + 3}\, dx$ (try $t = \tan\left(\frac{x}{2}\right)$)

40.) $\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(mx) \sin(nx)\, dx = \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$

41.) $\int \frac{x}{\sqrt{1-x}}\, dx$

Sol^{n}'s to selected extra credit problems

2.) $$\int e^{kx}\,dx = \int e^u \frac{du}{k} = \frac{1}{k}e^u + C = \boxed{\frac{1}{k}e^{kx} + C}$$ (used $u = kx$)

3.) $$\int_0^{\pi/2} \sec\theta \tan\theta \, d\theta = \lim_{t\to\frac{\pi}{2}^-} \int_0^t \sec\theta\tan\theta\,d\theta = \lim_{t\to\frac{\pi}{2}^-}\left(\sec(t) - \underset{1}{\sec(0)}\right) = \text{d.n.e}$$

(sec(0) undefined.)

4.) $$\int u^2(1+au^3)^7\,du = \frac{1}{3a}\int w^7\,dw$$

$w = 1 + au^3$
$dw = 3au^2du$
$u^2du = \frac{1}{3a}dw$

$$= \frac{1}{24a}w^8 + C$$
$$= \boxed{\frac{1}{24a}(1+au^3)^8 + C}$$

5.) $$\int \cos\theta\, e^{\sin\theta}\,d\theta = \int e^u du$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$$= e^u + C$$
$$= \boxed{e^{\sin\theta} + C}$$

6.) $$\int e^{\tan\theta}\sec^2\theta\,d\theta = \int e^u du = e^u + C = \boxed{e^{\tan\theta} + C}$$

$u = \tan\theta$
$du = \sec^2\theta\, d\theta$

7.) $$\int \frac{2t}{t^2-25}\,dt = \int \frac{du}{u}$$

$u = t^2 - 25$
$du = 2t\,dt$

$$= \ln|u| + C$$
$$= \boxed{\ln|t^2-25| + C}$$

8.) $$\int \frac{1}{\ln(x)}\frac{dx}{x} = \int \frac{1}{u}\,du$$

$u = \ln(x)$
$du = \frac{dx}{x}$

$$= \ln|u| + C$$
$$= \boxed{\ln|\ln(x)| + C}$$

9.) $$\int \cos(\ln(\theta) - A)\,\frac{d\theta}{\theta} = \int \cos(u)\,du \leftarrow \boxed{\begin{array}{l} u = \ln\theta - A \\ du = \frac{d\theta}{\theta} \end{array}}$$

$$= \sin(u) + C$$

$$= \boxed{\sin(\ln\theta - A) + C}$$

10.)

$$\int 2^{\tan(x)}\sec^2(x)\,dx = \int 2^u\,du \leftarrow \boxed{\begin{array}{l} u = \tan(x) \\ du = \sec^2(x)\,dx \end{array}}$$

$$= \frac{1}{\ln(2)}\,2^u + C$$

$$= \boxed{\frac{1}{\ln(2)}\,2^{\tan(x)} + C}$$

11.) $$\int x\,3^{x^2}\,dx = \int 3^u\,\frac{du}{2} \leftarrow \boxed{\begin{array}{l} u = x^2 \\ du = 2x\,dx \\ x\,dx = \frac{du}{2} \end{array}}$$

$$= \frac{1}{2\ln(3)}\,3^u + C$$

$$= \boxed{\frac{1}{2\ln(3)}\,3^{x^2} + C}$$

12.)

$$\int \frac{1}{v}\tan(\ln(v))\,dv = \int \tan(\theta)\,d\theta \leftarrow \boxed{\begin{array}{l} \theta = \ln(v) \\ d\theta = \frac{dv}{v} \end{array}}$$

$$= \int \frac{\sin\theta\,d\theta}{\cos\theta}$$

$$= \int \frac{-du}{u} \leftarrow \boxed{\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \end{array}}$$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\cos\theta| + C}$$

13.) $$\int \csc^2(1 + \ln(r))\,\frac{dr}{r} = \int \csc^2(u)\,du \leftarrow \boxed{\begin{array}{l} u = 1 + \ln(r) \\ du = \frac{dr}{r} \end{array}}$$

$$= -\cot(u) + C$$

$$= \boxed{-\cot(1 + \ln(r)) + C}$$

14.) $$\int_{-4}^{-3} \frac{1}{x}\,dx = \ln|x|\Big|_{-4}^{-3} \leftarrow \text{absolute value is key.}$$

$$= \ln|-3| - \ln|-4|$$

$$= \ln(3) - \ln(4) = \boxed{\ln(3/4)}$$

15.) $$\frac{d}{dx}\int_{\sin(x)}^{x} g(u)\,du = \frac{d}{dx}\big(G(x) - G(\sin(x))\big)$$

$$= G'(x) - G'(\sin(x))\cos(x)$$

$$= \boxed{g(x) - g(\sin(x))\cos(x)}$$

(G is antiderivative of g)

16. $$\int \frac{x\,dx}{6 - 3x} = \frac{1}{3}\int \frac{x}{2 - x}\,dx$$

$$= \frac{1}{3}\int \frac{x - 2 + 2}{2 - x}\,dx$$

$$= \frac{1}{3}\int \left(-1 + \frac{2}{2 - x}\right)dx$$

$$= \frac{1}{3}\int \left(-1 + \frac{2}{u}\right)(-du) \leftarrow \boxed{\begin{array}{l} u = 2 - x \\ du = -dx \end{array}}$$

$$= \frac{1}{3}\big(u - 2\ln|u|\big) + C$$

$$= \boxed{\frac{1}{3}\big(2 - x - 2\ln|2 - x|\big) + C}$$

17.) $$\int \underbrace{x}_{u}\,\underbrace{e^{3x}dx}_{dv} = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx \qquad \boxed{\begin{array}{l} u = x \\ du = dx \\ dv = e^{3x}dx \\ v = \frac{1}{3}e^{3x} \end{array}}$$

$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

$$= \boxed{\frac{1}{3}e^{3x}\left(x - \frac{1}{3}\right) + C}$$

18.) $$\int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta \qquad \left(\begin{array}{l}\text{recall} \\ \tan^2\theta = \sec^2\theta - 1\end{array}\right)$$

$$= \boxed{\tan\theta - \theta + C}$$

19.) $$\int \cos(\pi t + k)\,dt = \boxed{\frac{1}{\pi}\sin(\pi t + k) + C}$$

(Just let $u = \pi t + k$ to see it.)

20.) $$\int \frac{dx}{x^2 - 1} = \frac{1}{2}\int \left(\frac{1}{x - 1} - \frac{1}{x + 1}\right)dx \qquad \left(\begin{array}{l}\text{Some algebra} \\ \text{to do here}\end{array}\right)$$

$$= \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{dW}{W} \leftarrow \boxed{\begin{array}{l} u = x - 1 \\ W = x + 1 \end{array}}$$

$$= \frac{1}{2}\big(\ln|u| - \ln|W|\big) + C$$

$$= \boxed{\frac{1}{2}\big(\ln|x - 1| - \ln|x + 1|\big) + C}$$

21.) $$\frac{1}{x^2 + 5x + 6} = \frac{1}{(x + 3)(x + 2)} = \frac{A}{x + 3} + \frac{B}{x + 2}$$

$$1 = A(x + 2) + B(x + 3)$$

$x = -2$ | $1 = B$

$x = -3$ | $1 = -A \rightarrow A = -1$

Thus $$\frac{1}{x^2 + 5x + 6} = \frac{-1}{x + 3} + \frac{1}{x + 2}$$

$$\int \frac{dx}{x^2 + 5x + 6} = \int \frac{-1}{x + 3}\,dx + \int \frac{1}{x + 2}\,dx$$

$$= \boxed{-\ln|x + 3| + \ln|x + 2| + C}$$

22. $$\int \log_A(\theta)\,\frac{d\theta}{\theta} = \int \frac{\ln(\theta)}{\ln(A)}\,\frac{d\theta}{\theta}$$

$$= \frac{1}{\ln(A)}\int u\,du \qquad \boxed{\begin{aligned} u &= \ln(\theta) \\ du &= d\theta/\theta \end{aligned}}$$

$$= \frac{1}{2\ln(A)}\,u^2 + C$$

$$= \boxed{\frac{1}{2\ln(A)}(\ln\theta)^2 + C}$$

23. $$\int \frac{3\,dx}{4+3x^2} = \int \frac{dx}{4/3 + x^2} \qquad \boxed{\begin{aligned} x &= \tfrac{2}{\sqrt{3}}\tan\theta \\ dx &= \tfrac{2}{\sqrt{3}}\sec^2\theta\,d\theta \\ \tfrac{4}{3} + x^2 &= \tfrac{4}{3}\sec^2\theta \end{aligned}}$$

$$= \int \frac{2/\sqrt{3}\,\cancel{\sec^2\theta}\,d\theta}{(4/3)\cancel{\sec^2\theta}}$$

$$= \frac{\sqrt{3}}{2}\int d\theta$$

$$= \frac{\sqrt{3}}{2}\theta + C = \boxed{\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C}$$

24. $$\int e^\theta\cos\theta\,d\theta = e^\theta\sin\theta - \int e^\theta\sin\theta\,d\theta$$

$$= e^\theta\sin\theta + e^\theta\cos\theta - \int e^\theta\cos\theta\,d\theta$$

$$\Rightarrow \quad \int e^\theta\cos\theta\,d\theta = \boxed{\frac{1}{2}e^\theta(\sin\theta + \cos\theta) + C}$$

25. $$\int \cos^4\theta\,d\theta = \int\left[\frac{1}{2}(1+\cos(2\theta))\right]^2 d\theta$$

$$= \frac{1}{4}\int\left[1 + 2\cos(2\theta) + \cos^2(2\theta)\right]d\theta$$

$$= \frac{1}{4}\int\left[1 + 2\cos(2\theta) + \frac{1}{2}(1+\cos(4\theta))\right]d\theta$$

$$= \frac{1}{4}\int\left(\frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta)\right)d\theta$$

$$= \boxed{\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + C}$$

26. $$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{\cancel{3\cos\theta}\,d\theta}{\cancel{3\cos\theta}} \qquad \boxed{\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta\,d\theta \\ 9-x^2 &= 9\cos^2\theta \end{aligned}}$$

$$= \theta + C$$

$$= \boxed{\sin^{-1}(x/3) + C}$$

in the last step we solved for θ,

$$x = 3\sin\theta$$

$$\frac{x}{3} = \sin\theta$$

$$\sin^{-1}\left(\frac{x}{3}\right) = \sin^{-1}(\sin\theta) = \theta.$$

27. $$\int(\tan^{-1}(x)+1)\frac{dx}{1+x^2} = \int(u+1)\,du \qquad \boxed{\begin{aligned} u &= \tan^{-1}(x) \\ du &= \frac{dx}{1+x^2} \end{aligned}}$$

$$= \frac{1}{2}u^2 + u + C$$

$$= \boxed{\frac{1}{2}[\tan^{-1}(x)]^2 + \tan^{-1}(x) + C}$$

28. $$\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\left(-xe^{-x} - e^{-x}\Big|_0^t\right)$$

skipped - how to integrate xe^{-x}

$$= \lim_{t\to\infty}\left(-te^{-t} + \underset{\searrow 0}{\cancel{e^{-t}}} + 1\right)$$

$$= \lim_{t\to\infty}\left(\frac{-t}{e^t}\right) + 1$$

$$\underset{(\infty/\infty)}{\overset{LH}{=}} \lim_{t\to\infty}\left(\frac{-1}{e^t}\right)_{\searrow 0} + 1 = \boxed{1}$$

29. $$\int_0^\infty x^2e^{-x}\,dx = \lim_{t\to\infty}\left((-x^2-2x-2)e^{-x}\Big|_0^t\right)$$

$$= \lim_{t\to\infty}\left(\frac{-t^2-2t-2}{e^t} + 2\right)$$

$$\underset{(\infty/\infty)}{\overset{LH}{=}} \lim_{t\to\infty}\left(\frac{-2t-2}{e^t}\right) + 2$$

$$\underset{(\infty/\infty)}{\overset{LH}{=}} \lim_{t\to\infty}\left(\frac{-2}{e^t}\right)_{\searrow 0} + 2 = \boxed{2}$$

30. $$\int \underbrace{x^n}_{u}\,\underbrace{e^{-x}dx}_{dV} = -x^ne^{-x} + n\int x^{n-1}e^{-x}\,dx \qquad \begin{aligned} u &= x^n \\ du &= nx^{n-1}dx \\ dV &= e^{-x}dx \\ V &= -e^{-x} \end{aligned}$$

$$\int x^{n-1}e^{-x}\,dx = -x^{n-1}e^{-x} + (n-1)\int x^{n-2}e^{-x}\,dx$$

$$\vdots$$

$$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} \leftarrow$$ nested inside n - int. by parts. • think

$$\int x^ne^{-x}\,dx = \underbrace{(-x^n - \cdots - n!)}_{\text{clearly some polynomial from the work above.}}e^{-x} + C$$

$$\int_0^\infty x^ne^{-x}\,dx = \lim_{t\to\infty}\left(\frac{-t^n - \cdots - n!}{e^t}\right) + n!$$

$$= \boxed{n!}$$

(apply L'Hopital's rule n times just like 28 & 29.)

31.) $\int \sin\theta \, d\theta = \boxed{-\cos\theta + C}$

$\int \sin^2\theta \, d\theta = \int \frac{1}{2}(1-\cos(2\theta))\,d\theta$

$= \boxed{\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C}$

$\int \sin^3\theta \, d\theta = \int (1-\cos^2\theta)\sin\theta \, d\theta$

$= \int (u^2-1)\,du$ $\quad u = \cos\theta$, $du = -\sin\theta \, d\theta$

$= \boxed{\frac{1}{3}\cos^3\theta - \cos\theta + C}$

$\int \sin^4\theta \, d\theta = \int \left[\frac{1}{2}(1-\cos(2\theta))\right]^2 d\theta$

$= \int \frac{1}{4}\left(1 - 2\cos(2\theta) + \cos^2(2\theta)\right)d\theta$

$= \frac{1}{4}\int \left(1 - 2\cos(2\theta) + \frac{1}{2} + \frac{1}{2}\cos(4\theta)\right)d\theta$

$= \boxed{\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + C}$

$\int \sin^5\theta \, d\theta = \int (1-\cos^2\theta)^2 \sin\theta \, d\theta$ $\quad u = \cos\theta$

$= \int (1 - u^2)^2(-du)$

$= -\int (1 - 2u^2 + u^4)\,du$

$= \boxed{-\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C}$

$\int \sin^6\theta \, d\theta = \int \left[\frac{1}{2}(1-\cos(2\theta))\right]^3 d\theta$

$= \frac{1}{8}\int \left(1 - 3\cos(2\theta) + 3\cos^2(2\theta) - \cos^3(2\theta)\right)d\theta$

$= \frac{1}{8}\theta - \frac{3}{16}\sin(2\theta) +$

$+ \frac{3}{8}\int \frac{1}{2}(1+\cos(4\theta))\,d\theta +$

$- \frac{1}{8}\int (1-\sin^2(2\theta))\cos(2\theta)\,d\theta$

$$= \boxed{\begin{aligned}&\frac{1}{8}\theta - \frac{3}{16}\sin(2\theta) \\ &+ \frac{3}{16}\theta + \frac{3}{32}\sin(4\theta) \\ &- \frac{1}{16}\left(\sin(2\theta) - \frac{1}{3}\sin^3(2\theta)\right) + C\end{aligned}}$$

32.) $\int \sec\theta \, d\theta = \boxed{\ln|\sec\theta + \tan\theta| + C}$: See notes.

$\int \sec^2\theta \, d\theta = \boxed{\tan\theta + C}$: basic int.

$\int \underbrace{\sec\theta}_{u} \, \underbrace{\sec^2\theta \, d\theta}_{dV} = \sec\theta\tan\theta - \int \tan^2\theta \sec\theta \, d\theta$

$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta \, d\theta$

$= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| -$

$\int \sec^3\theta \, d\theta$

$\therefore \int \sec^3\theta \, d\theta = \boxed{\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C}$

$\int \sec^4\theta \, d\theta = \int (1+\tan^2\theta)\sec^2\theta \, d\theta$

$= \int (1+u^2)\,du$

$= \boxed{\tan\theta + \frac{1}{3}\tan^3\theta + C}$

33.) $\frac{x^2}{(x-1)(x-2)} = \frac{x^2}{x^2-3x+2}$

$$\begin{array}{r} 1 \\ x^2 \overline{)\,x^2-3x+2} \\ \underline{x^2} \\ 3x-2 \end{array} \Rightarrow \frac{x^2}{x^2-3x+2} = 1 + \frac{3x-2}{x^2-3x+2}$$

$\frac{3x-2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{-1}{x-1} + \frac{4}{x-2}$

Some work.

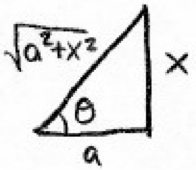
$\int \frac{x^2\,dx}{x^2-3x+2} = \int \left(1 - \frac{1}{x-1} + \frac{4}{x-2}\right)dx$

$= \boxed{x - \ln|x-1| + 4\ln|x-2| + C}$

34.) $\int \sqrt{a^2+x^2}\,dx = \int a^2\sec^3\theta \, d\theta$ $\quad x = a\tan\theta$, $dx = a\sec^2\theta \, d\theta$

$= \frac{a^2}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$

$= \boxed{\frac{a^2}{2}\left(\frac{x\sqrt{a^2+x^2}}{a^2} + \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|\right) + C}$



$\sec\theta = \frac{\sqrt{a^2+x^2}}{a}$

35.) $\int \sin(x)\cos(3x)\,dx = \int \frac{1}{2i}(e^{ix}-e^{-ix})\frac{1}{2}(e^{3ix}+e^{-3ix})\,dx$

$= \frac{1}{4i}\int\left(e^{4ix}+e^{-2ix}-e^{2ix}-e^{-4ix}\right)dx$

$= \int\left[\frac{1}{2}\frac{1}{2i}\left(e^{4ix}-e^{-4ix}\right)-\frac{1}{2}\frac{1}{2i}\left(e^{2ix}-e^{-2ix}\right)\right]dx$

$= \int\left(\frac{1}{2}\sin(4x)-\frac{1}{2}\sin(2x)\right)dx$

$= \boxed{\frac{-1}{8}\cos(4x)-\frac{1}{4}\cos(2x)+C}$

behold, the complex exponential!

42.) $\frac{dY}{dx} = xe^{Y} \Rightarrow e^{-Y}dY = x\,dx \Rightarrow -e^{-Y} = \frac{1}{2}x^2+\tilde{C}$

$\Rightarrow e^{-Y} = C-\frac{1}{2}x^2$

$\Rightarrow -Y = \ln(C-\frac{1}{2}x^2)$

$\Rightarrow \boxed{Y = -\ln(C-\frac{1}{2}x^2)}$

43.) $\frac{dY}{dx} = \sqrt{x}\,Y \Rightarrow \frac{dY}{Y} = \sqrt{x}\,dx \Rightarrow \ln|Y| = \frac{2}{3}x^{3/2}+C \Rightarrow \boxed{Y = \pm e^{\frac{2}{3}x^{3/2}+C}}$

44.) $Y''+13Y'+12Y = 0$

$\lambda^2+13\lambda+12 = 0 \Rightarrow \lambda = \frac{-13\pm\sqrt{169-48}}{2} = \frac{-13\pm 11}{2} = \frac{-2}{2} \text{ or } \frac{-24}{2} = -1 \text{ or } -12$

$\Rightarrow \boxed{Y = C_1e^{-x}+C_2e^{-12x}}$

45.) $Y''+13Y'+12Y = x^2$

$\Rightarrow Y_p = Ax^2+Bx+C$, $Y_p' = 2Ax+B$, $Y_p'' = 2A$ $\Rightarrow 2A+13(2Ax+B)+12(Ax^2+Bx+C) = x^2$

$2A+13B+12C = 0$
$26A+12B = 0$
$12A = 1$

Solve these to find
$A = 1/12$
$B = -13/72$
$C = 157/864$

$\boxed{Y = C_1e^{-x}+C_2e^{-12x}+\frac{1}{12}x^2-\frac{13}{72}x+\frac{157}{864}}$

46.) $Y'' = x^2 \Rightarrow Y' = \frac{1}{3}x^3+C \Rightarrow \boxed{Y = \frac{1}{12}x^4+Cx+d}$

just integrate twice.

47.) $Y''+4Y = \sin(x) \Rightarrow \lambda^2+4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow \underline{Y_c} = C_1\cos(2x)+C_2\sin(2x)$

$Y_p = A\sin(x)+B\cos(x)$
$Y_p' = A\cos(x)-B\sin(x)$
$Y_p'' = -Y_p$

$Y_p''+4Y_p = \sin(x)$
$3A\sin(x)+3B\cos(x) = \sin(x)$
$A = 1/3$
$B = 0$

$\boxed{Y = C_1\cos(2x)+C_2\sin(2x)+\frac{1}{3}\sin(x)}$

48.) $\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}}$

50.) $\boxed{x^2 + 3x - 2}$

51.) $\sin(x)\cos(x) = \frac{1}{2}\sin(2x) = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n+1}}{(2n+1)!}}$

52.) $5^x = e^{\ln(5^x)} = e^{x\ln(5)} = \sum_{n=0}^{\infty} \frac{(x\ln(5))^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(\ln(5))^n}{n!} x^n}$

53.) $\sqrt{\sin(x)} = f(\pi/2) + f'(\pi/2)(x-\pi/2) + \frac{1}{2}f''(\pi/2)(x-\pi/2)^2 + \cdots$

$= \sqrt{\sin \pi/2} + \frac{\cos(\pi/2)}{2\sqrt{\sin(\pi/2)}}(x-\pi/2) + \frac{1}{2}\left(\frac{-\sin^2(\pi/2)+1}{4(\sin(\pi/2))^{3/2}}\right)(x-\pi/2)^2 + \cdots$

$= \boxed{\frac{-1}{1536}(2x-\pi)^4 - \frac{1}{16}(2x-\pi)^2 + 1}$

54.) $\tan(x) = \tan(0) + \sec^2(0)\,x + \frac{1}{2}(2\tan(0)\sec^2(0))x^2 + \frac{1}{3!}\left[4\tan^2(0)\sec^2(0) + 2\sec^4(0)\right]x^3 + \cdots$

$= \boxed{x + \frac{1}{3}x^3 + \frac{2}{15}x^5} + \cdots$

55.) $\int \sqrt{\sin(x)}\,dx = \int\left[\frac{-1}{1536}(2x-\pi)^4 - \frac{1}{16}(2x-\pi)^2 + 1\right]dx$ (See how to finish ?)

56.) $\int \frac{\sin(x)}{x}\,dx = \int \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots}{x}\,dx$

$= \int\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^5 + \cdots\right)dx$

$= \boxed{x - \frac{1}{18}x^3 + \frac{1}{720}x^6 + C + \cdots}$