MA 241-003 FINAL EXAM : SHOW WORK ON SEPERATE PAPER. NAME :

Calculations done on the exam will not be graded. 12/9/05

(1) (60 pts.) Calculate the integrals below, choose 12

a.) $\int \cos^2(\theta)\,d\theta$
b.) $\int \ln(x)\,dx$
c.) $\int \tan(x)\,dx$
d.) $\int \tan^2\theta\,d\theta$
e.) $\int \sec\theta\,d\theta$
f.) $\int \frac{x}{x+1}\,dx$
g.) $\int \frac{1}{x^2+5x+6}\,dx$
h.) $\int \sqrt{4-x^2}\,dx$
i.) $\int x^5 \ln(x)\,dx$
j.) $\int (x^2-1)^{-3/2}\,dx$
k.) $\int \frac{\pi}{9+x^2}\,dx$
l.) $\int (1+\sin\theta)^{10}\cos\theta\,d\theta$
m.) $\int (\cos^4\theta - \sin^4\theta)\,d\theta$
n.) $\int \sqrt{9+x^2}\,dx$
o.) $\int \sin^5\theta\,d\theta$

(2) (10 pts.) Calculate the Improper Integrals below, show limits involved explicitly.

a.) $\int_0^1 \frac{1}{x^2}\,dx$
b.) $\int_0^\infty e^{-x}\,dx$

(3) (20 pts.) Solve the DEq^{n}'s below as instructed

a.) $\frac{dy}{dx} = e^{x-y}$: find general sol^{n} (explicitly) (5pts.)
b.) $\frac{dy}{dx} = e^{\cos(x)}\sin(x)$: find all equilbrium sol^{n}'s (5pts.)
c.) $y'' + 5y' + 6y = x^2$: find general sol^{n} (10pts.)

(4) (20pts.) Use the known Maclaurin series or binomial series to calculate

a.) $\int \frac{\sin(x)}{x}\,dx$: find the power series sol$^{\underline{n}}$. (in Σ notation)

b.) $\left(\frac{x}{1+x^3}\right)^{10}$: find first 3 non-zero terms in the power series expansion about zero for the given function.

c.) xe^{-x} : find first 4 non-zero terms in the Maclaurin series.

d.) $\cos^4\theta$: find the power series representation of this function.

We derived the Maclaurin series in lecture:

$$\sin(u) = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$$

$$\cos(u) = 1 - \frac{1}{2}u^2 + \frac{1}{4!}u^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!}$$

$$\exp(u) = 1 + u + \frac{1}{2}u^2 + \frac{1}{3!}u^3 + \cdots = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

$$(1+u)^k = 1 + ku + \frac{1}{2}k(k-1)u^2 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} u^n$$

where $\binom{k}{n} \equiv \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$

Have a merry Christmas and a Happy New Year, enjoy the break, see you around, stop by if you ever have a question...

1 a.) $\int \cos^2\theta\, d\theta = \int \frac{1}{2}(1+\cos(2\theta))\,d\theta = \boxed{\frac{\theta}{2} + \frac{1}{4}\sin(2\theta) + C}$

b.) $\int \ln(x)\,dx \underset{\text{IBP}}{=} x\ln(x) - \int x\,\frac{dx}{x} = \boxed{x\ln(x) - x + C}$

c.) $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{-du}{u} = -\ln|u| + C = \boxed{-\ln|\cos\theta| + C}$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$

d.) $\int \tan^2\theta\, d\theta = \int(\sec^2\theta - 1)\,d\theta = \boxed{\tan\theta - \theta + C}$

e.) $\int \sec\theta\, d\theta = \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sec\theta + \tan\theta| + C}$

$u = \sec\theta + \tan\theta$
$du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta = \sec\theta(\sec\theta + \tan\theta)\,d\theta \Rightarrow \underline{\frac{du}{u} = \sec\theta\, d\theta}$

f.) $\int \frac{x}{x+1}\,dx = \int\left(\frac{x+1-1}{x+1}\right)dx = \int\left(1 - \frac{1}{x+1}\right)dx = \boxed{x - \ln|x+1| + C}$

g.) $\int \frac{1}{x^2+5x+6}\,dx = \int\left(\frac{-1}{x+3} + \frac{1}{x+2}\right)dx = \boxed{-\ln|x+3| + \ln|x+2| + C}$

$$\frac{1}{x^2+5x+6} = \frac{1}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + B(x+3)$$

$\underline{x=-2}$ $1 = B$
$\underline{x=-3}$ $1 = -A$ $\therefore A = -1$

h.) $\int\sqrt{4-x^2}\,dx = \int\sqrt{4\cos^2\theta}\; 2\cos\theta\, d\theta$ ← $\boxed{\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta\, d\theta \\ 4 - x^2 = 4\cos^2\theta \end{array}}$ $\Rightarrow \theta = \sin^{-1}\left(\frac{x}{2}\right)$

$= 4\int \cos^2\theta\, d\theta$

$= 2\int(1+\cos(2\theta))\,d\theta$

$= 2\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$

$= \boxed{2\left(\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{x}{2}\right)\right)\right) + C}$

① Continued

i.) $\int x^5 \ln(x)\,dx \underset{IBP}{=} \frac{1}{6}x^6\ln(x) - \int \frac{1}{6}x^6 \frac{dx}{x} = \boxed{\frac{1}{6}x^6\ln(x) - \frac{1}{36}x^6 + c}$

j.) $\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan^3\theta}$ ← $\boxed{\begin{array}{l} x = \sec\theta \\ x^2-1 = \sec^2\theta - 1 = \tan^2\theta \\ (\tan^2\theta)^{3/2} = \tan^3\theta \\ dx = \sec\theta\tan\theta\,d\theta \end{array}}$

$= \int \frac{\cos\theta\,d\theta}{\sin^2\theta}$

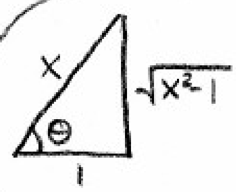
$= \int \frac{du}{u^2}$ ← $\left\{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}\right\}$

$= \frac{-1}{u} + c$

$= \frac{-1}{\sin\theta} + c$

$= \boxed{\frac{-x}{\sqrt{x^2-1}} + c}$

$x = \sec\theta$



k.) $\int \frac{\pi}{9+x^2}\,dx = \int \frac{\pi\, 3\cancel{\sec^2\theta}\,d\theta}{9\cancel{\sec^2\theta}}$ ← $\boxed{\begin{array}{l} x = 3\tan\theta \\ 9+x^2 = 9+9\tan^2\theta = 9\sec^2\theta \\ dx = 3\sec^2\theta\,d\theta \end{array}}$

$= \frac{\pi}{3}\int d\theta$

$= \frac{\pi}{3}\theta + c$

$= \boxed{\frac{\pi}{3}\tan^{-1}\left(\frac{x}{3}\right) + c}$

l.) $\int (1+\sin\theta)^{10}\cos\theta\,d\theta = \int (1+u)^{10}\,du$ ← $\boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}}$

$= \int w^{10}\,dw$ ← $\boxed{\begin{array}{l} w = 1+u \\ dw = du \end{array}}$

$= \frac{1}{11}w^{11} + c$

$= \frac{1}{11}(1+u)^{11} + c$

$= \boxed{\frac{1}{11}(1+\sin\theta)^{11} + c}$

① continued

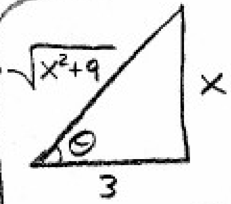
m.) $$\int(\cos^4\theta - \sin^4\theta)\,d\theta = \int(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)\,d\theta$$
$$= \int(\cos^2\theta - \sin^2\theta)\,d\theta$$
$$= \int\left[\tfrac{1}{2}(1+\cos(2\theta)) - \tfrac{1}{2}(1-\cos(2\theta))\right]d\theta$$
$$= \int \cos(2\theta)\,d\theta$$
$$= \boxed{\tfrac{1}{2}\sin(2\theta) + C}$$

n.) $$I = \int\sqrt{9+x^2}\,dx = \int 9\sec^3\theta\,d\theta$$

$$\boxed{\begin{aligned} x &= 3\tan\theta \\ \sqrt{9+x^2} &= \sqrt{9\sec^2\theta} = 3\sec\theta \\ dx &= 3\sec^2\theta\,d\theta \end{aligned}}$$

$$= 9\int \underbrace{\sec\theta}_{u}\ \underbrace{\sec^2\theta\,d\theta}_{dv}$$
$$\underset{\text{IBP}}{=} 9\left[\sec\theta\tan\theta - \int\tan^2\theta\sec\theta\,d\theta\right]$$
$$\underset{\text{I.B.P.}}{=} 9\left[\sec\theta\tan\theta - \int(\sec^2\theta - 1)\sec\theta\,d\theta\right]$$
$$= 9\left[\sec\theta\tan\theta - \int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta\right]$$
$$= 9\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int\sec^3\theta\,d\theta\right]$$
$$= 9(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) - \int 9\sec^3\theta\,d\theta$$

$$\Rightarrow I = \tfrac{9}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$
$$= \boxed{\tfrac{9}{2}\left(\frac{x\sqrt{x^2+9}}{9} + \ln\left|\tfrac{1}{3}\sqrt{x^2+9} + \tfrac{1}{3}x\right|\right) + C}$$



$x = 3\tan\theta$

$\tan\theta = \frac{x}{3}$

$\sec\theta = \frac{\sqrt{x^2+9}}{3}$

$\tan\theta = \frac{x}{3}$

o.) $$\int\sin^5\theta\,d\theta = \int(1-\cos^2\theta)^2\sin\theta\,d\theta$$
$$= -\int(1-u^2)^2\,du \qquad \begin{aligned} u &= \cos\theta \\ du &= -\sin\theta\,d\theta \end{aligned}$$
$$= -\int(1 - 2u^2 + u^4)\,du$$
$$= -u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5 + C = \boxed{-\cos\theta + \tfrac{2}{3}\cos^3\theta - \tfrac{1}{5}\cos^5\theta + C}$$

② a.) $\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\left(\int_t^1 \frac{1}{x^2}\,dx\right) = \lim_{t\to 0^+}\left(\frac{-1}{1} + \underset{\infty}{\frac{1}{t}}\right) = \boxed{\infty}$

b.) $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\left(\int_0^t e^{-x}\,dx\right) = \lim_{t\to\infty}\left(-\underset{0}{e^{-t}} + 1\right) = \boxed{1}$

③ a.) $\frac{dy}{dx} = e^{x-y} = e^x/e^y \Rightarrow \int e^y\,dy = \int e^x\,dx$
$\Rightarrow e^y = e^x + C$
$\Rightarrow \boxed{y = \ln(e^x + c)}$

b.) $\frac{dy}{dx} = e^{\cos(x)}\sin(x) = 0$ for eq. sol^{n}'s.

$e^{\cos(x)} > 0 \Rightarrow \sin(x) = 0 \Rightarrow \boxed{x = n\pi,\ n \in \mathbb{Z} = 0, \pm\pi, \pm 2\pi, \ldots}$

c.) $y'' + 5y' + 6y = x^2$

$\lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0 \Rightarrow \boxed{y_c = c_1 e^{-3x} + c_2 e^{-2x}}$

No overlap $\Rightarrow$ $y_p = Ax^2 + Bx + C$
$y_p' = 2Ax + B$
$y_p'' = 2A$

$\therefore\ 2A + 5(2Ax + B) + 6(Ax^2 + Bx + C) = x^2$

$(2A + 5B + 6C) + (10A + 6B)x + (6A)x^2 = x^2$

$6A = 1 \quad \therefore\ A = 1/6$

$10A + 6B = \frac{10}{6} + 6B = 0 \quad \therefore\ B = \frac{-10}{36}$

$2A + 5B + 6C = \frac{2}{6} - \frac{50}{36} + 6C = 0 \quad \therefore\ C = \left(\frac{50}{36} - \frac{12}{36}\right)\frac{1}{6}$
$= \frac{38}{216} = C$

$\therefore\ \boxed{y_{gen} = c_1 e^{-3x} + c_2 e^{-2x} + \frac{1}{6}x^2 - \frac{5}{18}x + \frac{38}{216}}$

(4) a.) $\dfrac{\sin(x)}{x} = \dfrac{1}{x}\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n}}{(2n+1)!}$

$$\therefore \int \frac{\sin(x)}{x}\,dx = \int\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}\right)dx$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}\int x^{2n}\,dx$$
$$= \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}\frac{1}{(2n+1)}x^{2n+1} + C}$$

b.) $\left(\dfrac{x}{1+x^3}\right)^{10} = x^{10}(1+x^3)^{-10}$

$$= x^{10}(1+u)^k$$
$$= x^{10}\left(1 + ku + \tfrac{1}{2}k(k-1)u^2 + \cdots\right)$$
$$= x^{10}\left(1 - 10x^3 + \tfrac{1}{2}(-10)(-11)\,9x^6 + \cdots\right)$$
$$= \boxed{x^{10} - 10x^{13} + 495x^{16} + \cdots}$$

c.) $xe^{-x} = x\left(1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \cdots\right)$

$$= \boxed{x - x^2 + \tfrac{1}{2}x^3 - \tfrac{1}{6}x^4 + \cdots}$$

d.) $\cos^4\theta = \cos^2\theta\,\cos^2\theta$

$$= \tfrac{1}{2}(1+\cos(2\theta))\,\tfrac{1}{2}(1+\cos(2\theta))$$
$$= \tfrac{1}{4}\left(1 + 2\cos(2\theta) + \cos^2(2\theta)\right)$$
$$= \tfrac{1}{4}\left(1 + 2\cos(2\theta) + \tfrac{1}{2}(1+\cos(4\theta))\right)$$
$$= \tfrac{3}{8} + \tfrac{1}{2}\cos(2\theta) + \tfrac{1}{8}\cos(4\theta)$$
$$= \tfrac{3}{8} + \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(2\theta)^{2n}}{(2n)!} + \tfrac{1}{8}\sum_{n=0}^{\infty}(-1)^n\frac{(4\theta)^{2n}}{(2n)!}$$
$$= \boxed{\tfrac{3}{8} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left[\cdot 2^{2n-1} + 2^{4n-3}\right]\theta^{2n}}$$

$4^{2n} = (2^2)^{2n} = 2^{4n}$

$\tfrac{1}{8} = 2^{-3}$