MA 241-003 – TEST II (75pts. in-class)

USE PLENTY OF PAPER, SHOW WORK CLEARLY, THANKS

10pts ① a.) $\int_0^1 \sqrt{x}\,dx$

b.) $\int_1^\infty \frac{\ln(x)}{x^3}\,dx$

② Consider the region bounded by $y = x$ and $y = x^2$.

36pts
a.) find the area of the region.
b.) find volume of solid obtained by rotating region about x-axis
c.) find volume of solid obtained by rot. region about y-axis
d.) find volume of solid obtained by rot. region about $y = 2$.

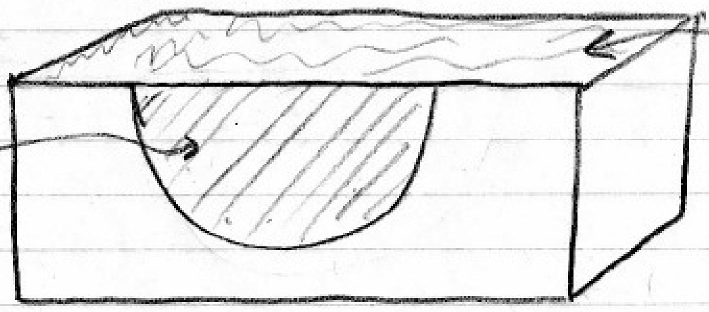
③ Find arclength of the parametric curve, $0 \le \theta \le \pi$

14pts
$$x = \sin\theta + \cos\theta$$
$$y = \sin\theta - \cos\theta$$

Hint: $\sin^2\theta + \cos^2\theta = 1$.

④ Find the hydrostatic force on the half-circle gate of radius R below. Assume water is all the way to the top of gate and water density is ρ.

15pts



MA 241 - 003 TEST II Sol^{n}

① a.) $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \boxed{\frac{2}{3}}$

b.) $\int \frac{\ln(x)}{x^3}dx = \frac{-\ln(x)}{2x^2} + \int \frac{1}{2x^2}\frac{dx}{x}$ ← IBP

$u = \ln(x)$	$dv = \frac{dx}{x^3}$
$du = \frac{dx}{x}$	$v = \frac{-1}{2x^2}$

$= \frac{-\ln(x)}{2x^2} + \frac{1}{2}\int \frac{1}{x^3}dx$

$= -\frac{1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4}\frac{1}{x^2} + C$

Ok, now do the improp. integral

$$\int_1^\infty \frac{\ln(x)}{x^3}dx = \lim_{t\to\infty} \int_1^t \left(\frac{\ln(x)}{x^3}\right)dx$$

$$= \lim_{t\to\infty} \left(\frac{-1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4}\frac{1}{x^2}\Big|_1^t\right.$$

$$= \lim_{t\to\infty}\left(\frac{-1}{2}\frac{\ln(t)}{t^2} - \underbrace{\frac{1}{4t^2}}_{\to 0} + \frac{1}{2}\frac{\overset{\to 0}{\ln(1)}}{1^2} + \frac{1}{4}\right)$$

$$= \frac{-1}{2}\lim_{t\to\infty}\left(\frac{\ln(t)}{t^2}\right) + \frac{1}{4}$$

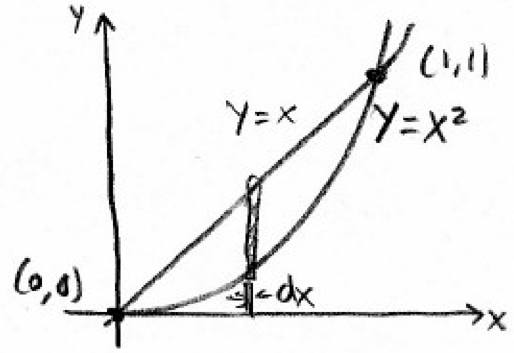
$$\overset{\mathcal{L}}{\underset{(\frac{\infty}{\infty})}{=}} \frac{-1}{2}\lim_{t\to\infty}\left(\frac{1/t}{2t}\right) + \frac{1}{4}$$

$$= \frac{-1}{4}\lim_{t\to\infty}\underbrace{\left(\frac{1}{t^2}\right)}_{\to 0} + \frac{1}{4}$$

$$= \boxed{\frac{1}{4}}$$

② Consider region bounded by $Y = X$ and $Y = X^2$

a.)



$$dA = (x - x^2)dx$$

$$A = \int_0^1 (x - x^2)\,dx$$

$$= \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right|_0^1$$

$$= \tfrac{1}{2} - \tfrac{1}{3}$$

$$= \boxed{1/6}$$

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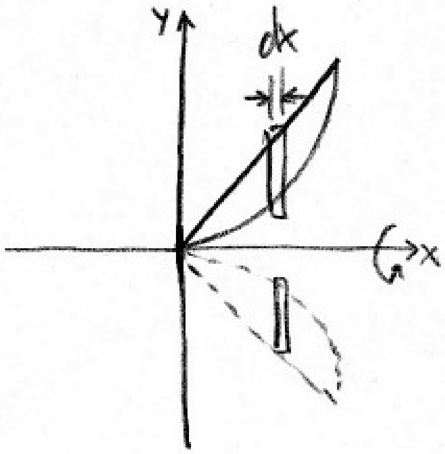
b.)



$r_{in} = x^2$ $\quad A = \pi(r_{out}^2 - r_{in}^2)$

$r_{out} = x$ $\quad dV = A\,dx$

$$dV = \pi(x^2 - x^4)\,dx$$

$$V = \int_0^1 \pi\,(x^2 - x^4)\,dx$$

$$= \pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right|_0^1$$

$$= \pi\left(\tfrac{1}{3} - \tfrac{1}{5}\right)$$

$$= \boxed{\frac{2\pi}{15}}$$

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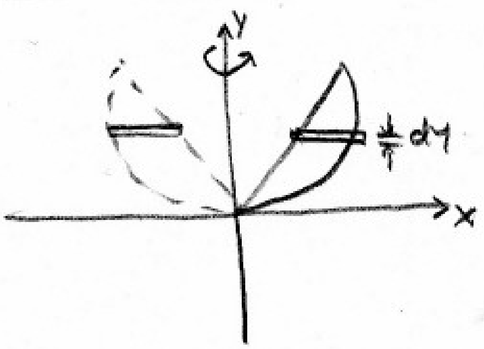
c.)



$r_{in} = X_L = Y$

$r_{out} = X_R = \sqrt{Y}$

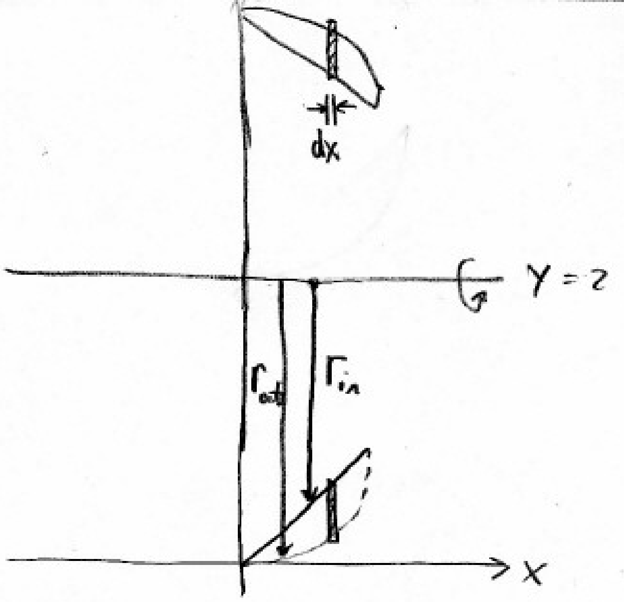
$$A = \pi(Y - Y^2)$$

$$dV = \pi(Y - Y^2)\,dY$$

$$V = \int_0^1 \pi(Y - Y^2)\,dY = \pi\left(\tfrac{Y^2}{2} - \tfrac{Y^3}{3}\right)\Big|_0^1$$

$$= \pi\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \boxed{\pi/6}$$

② d.)



$$r_{out} = 2 - x^2$$
$$r_{in} = 2 - x$$

$$A = \pi\left(r_{out}^2 - r_{in}^2\right)$$
$$= \pi\left((2-x^2)^2 - (2-x)^2\right)$$
$$= \pi\left(4 - 4x^2 + x^4 - 4 + 4x - x^2\right)$$
$$= \pi\left(x^4 - 5x^2 + 4x\right)$$

$$dV = A\,dx \quad \therefore \quad V = \int_0^1 \pi\left(x^4 - 5x^2 + 4x\right)dx$$
$$= \pi\left(\tfrac{1}{5}x^5 - \tfrac{5}{3}x^3 + 2x^2\right)\Big|_0^1$$
$$= \pi\left(\tfrac{1}{5} - \tfrac{5}{3} + 2\right)$$
$$= \pi\left(\frac{6 - 50 + 60}{30}\right) = \frac{16\pi}{30} = \boxed{\frac{8\pi}{15}}$$

familar ?

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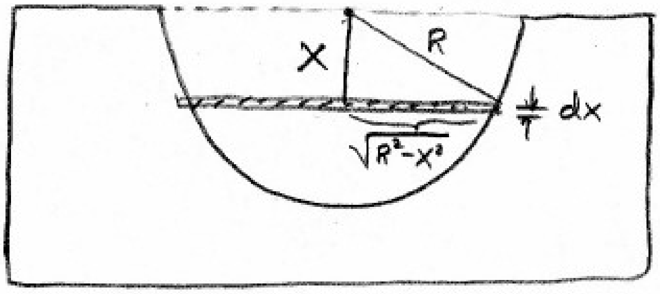
③ $x = \sin\theta + \cos\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta - \sin\theta$

$y = \sin\theta - \cos\theta \Rightarrow \frac{dy}{d\theta} = \cos\theta + \sin\theta$

$$S = \int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$
$$= \int_0^{\pi}\sqrt{(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2}\,d\theta$$
$$= \int_0^{\pi}\sqrt{\cos^2\theta - \cancel{2\sin\theta\cos\theta} + \sin^2\theta + \cos^2\theta + \cancel{2\sin\theta\cos\theta} + \sin^2\theta}\,d\theta$$
$$= \int_0^{\pi}\sqrt{2(\cos^2\theta + \sin^2\theta)}\,d\theta$$
$$= \int_0^{\pi}\sqrt{2}\,d\theta = \sqrt{2}\,\theta\Big|_0^{\pi} = \boxed{\pi\sqrt{2}}$$

(4) Find hydrostatic force on half-circle gate



$$dA = 2\sqrt{R^2-x^2}\,dx$$

$$P = \rho g x \quad \left(\begin{matrix}\text{x is same}\\ \text{as depth here.}\end{matrix}\right)$$

Now $P = \dfrac{dF}{dA}$ $\therefore$ $dF = P\,dA = \rho g x \cdot 2\sqrt{R^2-x^2}\,dx$

now sum the forces for all the strips on

$0 \le x \le R$,

$$F_{total} = \int_0^R \rho g \sqrt{R^2-x^2}\; 2x\,dx$$

$$\boxed{\begin{aligned} u &= R^2 - x^2 \\ du &= -2x\,dx \\ u(0) &= R^2 \\ u(R) &= 0 \end{aligned}}$$

$$= \int_{R^2}^{0} \rho g \left(\sqrt{u}\right)(-du)$$

$$= \rho g \int_0^{R^2} \sqrt{u}\,du$$

$$= \frac{2\rho g}{3} u^{\frac{3}{2}} \Big|_0^{R^2}$$

$$= \frac{2}{3}\rho g \left[(R^2)^{\frac{3}{2}} - (0)^{\frac{3}{2}}\right] \qquad (R^2)^{\frac{3}{2}} = R^3$$

$$= \boxed{\frac{2}{3}\rho g R^3}$$

SOLUTION TO TAKE-HOME PORTION (25%) OF TEST II

a.) $$\int \frac{dx}{1+9x^2} = \int \frac{\frac{1}{3}du}{1+u^2} = \frac{1}{3}\tan^{-1}(u) + c = \frac{1}{3}\tan^{-1}(3x) + C$$

$\boxed{\begin{array}{c}\text{used.}\\ u = 3x \\ \frac{du}{3} = dx\end{array}}$

$$\int_{-\infty}^{\infty} \frac{dx}{1+9x^2} = \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{1+9x^2} + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{dx}{1+9x^2}$$

$$= \lim_{t_1 \to -\infty} \left(\frac{1}{3}\tan^{-1}(3x) \Big|_{t_1}^{0} \right) + \lim_{t_2 \to \infty} \left(\frac{1}{3}\tan^{-1}(3x) \Big|_{0}^{t_2} \right)$$

$$= \lim_{t_1 \to -\infty} \left(\frac{1}{3} \cancelto{0}{\tan^{-1}(0)} - \frac{1}{3}\tan^{-1}(3t_1) \right) +$$

$$+ \lim_{t_2 \to \infty} \left(\frac{1}{3}\tan^{-1}(3t_2) - \frac{1}{3}\cancelto{0}{\tan^{-1}(0)} \right)$$

$$= -\frac{1}{3} \lim_{t_1 \to -\infty} \left(\tan^{-1}(3t_1) \right) + \frac{1}{3} \lim_{t_2 \to \infty} \left(\tan^{-1}(3t_2) \right)$$

$$= -\frac{1}{3}\left(\frac{-\pi}{2}\right) + \frac{1}{3}\left(\frac{\pi}{2}\right)$$

$$= \boxed{\pi/3}$$

Let $u = 3t$ to see $\lim_{t \to \infty} \left(\tan^{-1}(3t) \right) = \lim_{u \to \infty} \left(\tan^{-1}(u) \right) = \frac{\pi}{2}$, the factor of three doesn't change the limit.

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b.) As we have shown previously $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$

$$\int_0^{\pi/2} \sec(\theta)\,d\theta = \lim_{t \to (\pi/2)^-} \left(\int_0^t \sec(\theta)\,d\theta \right)$$

$$= \lim_{t \to \left(\frac{\pi}{2}\right)^-} \left(\ln|\sec(t) + \tan(t)| - \ln|\sec(0) + \tan(0)| \right)$$

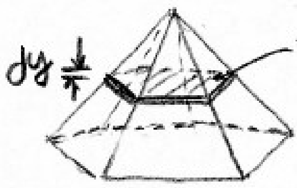
$$= \ln\left| \lim_{t \to (\pi/2)^-} \left(\frac{1 + \sin(t)}{\cos(t)} \right) \right| - \cancelto{0}{\ln(1+0)}$$

$$= \ln(\infty)$$

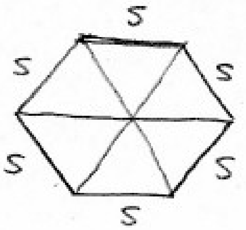
$$= \boxed{\infty}$$

Since $\cos(t) \longrightarrow$ small positive # as $t \to \frac{\pi}{2}$ from left. $(1+\sin(t)) \longrightarrow 2$. So we have $\frac{2}{\text{small } \# > 0} \longrightarrow \infty$.

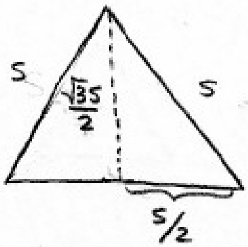
(2) Find volume of regular 6-sided pyramid of height h and side A.



typical slice of volume dV, with side s, height y from base with thickness dy.



A regular hexagon is composed of 6 equilateral triangles.
$A_{hex} = 6A_{\Delta}$



area of triangle is (base)(height)$\cdot\frac{1}{2}$

$$A_{\Delta} = \frac{1}{2}(s)\left(\frac{\sqrt{3}s}{2}\right) = \frac{\sqrt{3}}{4}s^2 \Rightarrow A_{hex} = \frac{6\sqrt{3}}{4}s^2 = \frac{3\sqrt{3}}{2}s^2$$

Thus we find the volume of typical slice is $dV = \frac{3\sqrt{3}}{2}s^2\,dy$.
Notice s is a linear function of y, $s(y) = my + b$,

$$\left.\begin{aligned} s(0) &= m(0) + b = A \quad \therefore\ b = A \\ s(h) &= m(h) + A = 0 \quad \therefore\ m = -A/h \end{aligned}\right\} \rightarrow s = A(1 - y/h)$$

Therefore, $dV = \frac{3\sqrt{3}}{2}A^2(1 - y/h)^2\,dy$

$$\begin{aligned} V &= \int_0^h \frac{3\sqrt{3}}{2}A^2(1 - y/h)^2\,dy \\ &= \frac{3\sqrt{3}}{2}A^2 \int_0^h \left(1 - \frac{2}{h}y + \frac{1}{h^2}y^2\right)dy \\ &= \frac{3\sqrt{3}}{2}A^2\left(y - \frac{1}{h}y^2 + \frac{1}{3h^2}y^3\right)\Bigg|_0^h \\ &= \frac{3\sqrt{3}}{2}A^2\left(\cancel{h} - \frac{1}{\cancel{h}}\cancel{h^2} + \frac{1}{3h^2}h^3\right) \\ &= \boxed{\frac{\sqrt{3}}{2}A^2h = V} \end{aligned}$$

③ a.) Given a particle of mass m is under the influence of a force F in the x- direction then (particle only moves in x- direction)

$$W_{ab} = \int_a^b F(x)\,dx$$

$$= \int_a^b ma\,dx \qquad \text{(assuming } F = ma\text{)}$$

$$= \int_a^b m \frac{dv}{dt}\,dx$$

$$= \int_a^b m \frac{dx}{dt}\frac{dv}{dx}\,dx \qquad \text{(Chain Rule)}$$

$$= \int_a^b m v \frac{dv}{dx}\,dx$$

$$= \int_{V_a}^{V_b} m v\,dV$$

(V- substitution, notice had to change bounds, $V_a = V(x=a)$)

$$= \frac{1}{2} m v^2 \Big|_{V_a}^{V_b}$$

$$= \frac{1}{2} m V_b^2 - \frac{1}{2} m V_a^2$$

$$= \boxed{T_b - T_a = W_{ab}}$$

← Work- Energy Thm. Notice this holds for all kinds of forces, friction, spring, ...

b.) $$W_{0 \to x_f} = \int_0^{x_f} (kx + mg)\,dx$$

k, m, and g are constants

$$= \left(\frac{1}{2}kx^2 + mgx\right|_0^{x_f}$$

$$= \boxed{\frac{1}{2}k x_f^2 + mg x_f}$$