MA 241-003 : TEST III - IN CLASS - DIFFERENTIAL EQ^{n}'s - 75pts + Bonus

(1.) Given that $\boxed{\frac{dy}{dx} = \cos(x)\sqrt{y}}$. Find then,

(24pts.) a.) ALL equilibrium sol^{n}'s.

b.) the general sol^{n} (explicitly)

c.) orthogonal trajectories (explicitly)

d.) the specific sol^{n} that passes thru (0,0).

(2.) Find the general sol^{n} to the DEq^{n} below

$$\boxed{y'' + 5y' + 6y = 12e^{x} + 6x + 11}$$

(27pts.)

(3.) For each of the DEq^{n}'s below find the complementary (aka homogeneous) sol^{n} and state the correct starting point for finding the particular sol^{n}, I mean state y_p but do not determine the unknown coefficients A, B, ...

(24pts.)

a.) $y'' + 2y' + 10y = x^2 + x\cos(x)$

b.) $y'' + 2y' + y = e^{-x}$

c.) $y'' + 9y = \cos(3x) - 6$

d.) $y'' + 8y' + 12y = e^{-2x} + 7x$

e.) $y'' + 16y = xe^{x}\sin(x)$

You may choose 4 out of these. give minimal answers (can't guess everything ☺)

Bonus: Use the ideas we've developed for 2nd order DEq^{n}'s to solve the first order differential eq^{n} below (α some constant.)

$$\boxed{\frac{dy}{dx} - 3y = 13e^{\alpha x}}$$

Notice that you cannot just seperate variables here.

Test III - in class part - sol^{n}

(1) $\frac{dy}{dx} = \cos(x)\sqrt{y}$

a.) $\frac{dy}{dx} = \cos(x)\sqrt{y} = 0 \Rightarrow \cos(x) = 0$ or $\sqrt{y} = 0$

$\Rightarrow$ $\boxed{x = n\pi + \frac{\pi}{2},\ n \in \mathbb{Z} \text{ and } y = 0 \text{ are the eq. sol}^{n}\text{'s.}}$

b.) $\int \frac{dy}{\sqrt{y}} = \int \cos(x)\,dx$

$2\sqrt{y} = \sin(x) + C$

$\sqrt{y} = \frac{1}{2}(\sin(x) + C)$ $\therefore$ $\boxed{y = \frac{1}{4}(\sin(x) + C)^2}$

c.) $\frac{dy}{dx} = \frac{-1}{\cos(x)\sqrt{y}}$ gives the O.T., so solve it.

$\int -\sqrt{y}\,dy = \int \frac{1}{\cos(x)}\,dx$

$-\int \sqrt{y}\,dy = -\frac{2}{3}y^{3/2} + C_1$

$\int \frac{1}{\cos(x)}\,dx = \int \sec(x)\,dx = \int \frac{du}{u}$

$= \ln|u| + C_2$

$= \ln|\sec(x) + \tan(x)| + C_2$

$u = \sec(x) + \tan(x)$

$\frac{du}{dx} = \sec(x)\tan(x) + \sec^2(x)$

$\Rightarrow du = \sec(x)\cdot u\,dx$

$\therefore \frac{du}{u} = \sec(x)\,dx$

Hence equating the integrals,

$-\frac{2}{3}y^{3/2} = \ln|\sec(x) + \tan(x)| + C_3$

$\Rightarrow \boxed{y = \left(-\frac{3}{2}\ln|\sec(x) + \tan(x)| + C_4\right)^{2/3}}$

d.) $y = 0$ when $x = 0$ should fix the value of C using the result of b.)

$0 = \frac{1}{4}(\sin(0) + C)^2 = \frac{1}{4}C^2 \Rightarrow \underline{C = 0}$

$\therefore \boxed{y = \frac{1}{4}\sin^2(x)}$

(2.) $y'' + 5y' + 6y = 12e^x + 6x + 11$

to begin find y_c.

$\lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0 \quad \therefore \lambda_1 = -3, \lambda_2 = -2$

$\Rightarrow \underline{y_c = c_1 e^{-3x} + c_2 e^{-2x}}$

now we find the particular sol^n using the method of undetermined coefficients. Begin with the educated guess

$$y_p = Ae^x + Bx + C$$
$$y_p' = Ae^x + B$$
$$y_p'' = Ae^x$$

$$y_p'' + 5y_p' + 6y_p = 12e^x + 6x + 11$$
$$Ae^x + 5(Ae^x + B) + 6(Ae^x + Bx + C) = 12e^x + 6x + 11$$
$$e^x(A + 5A + 6A) + x(6B) + 5B + 6C = e^x(12) + x(6) + 11$$

Equate Coefficients of e^x, x and constants,

e^x : $12A = 12 \Rightarrow A = 1$

x : $6B = 6 \Rightarrow B = 1$

x^0 : $5B + 6C = 11 \Rightarrow 6C = 11 - 5 = 6 \Rightarrow C = 1$

So we find the general sol^n $y_g = y_c + y_p$ is

$$\boxed{y_g = c_1 e^{-3x} + c_2 e^{-2x} + e^x + x + 1}$$

③ a.) $y'' + 2y' + 10y = x^2 + x\cos(x)$

$\lambda^2 + 2\lambda + 10 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$

$\Rightarrow \boxed{y_c = e^{-x}(c_1\cos(3x) + c_2\sin(3x))}$.

Then we guess

$\boxed{y_p = Ax^2 + Bx + C + x(D\cos(x) + E\sin(x)) + F\cos(x) + G\sin(x)}$

no overlap so that'll do.

b.) $y'' + 2y' + y = e^{-x}$

$\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0 \quad \therefore \lambda = -1$ twice

$\Rightarrow \boxed{y_c = c_1 e^{-x} + c_2 x e^{-x}}$

naively $y_p^n = Ae^{-x}$ (overlaps)

less naive $y_p^n = Axe^{-x}$ (still overlaps)

correct $\boxed{y_p = Ax^2 e^{-x}}$

c.) $y'' + 9y = \cos(3x) - 6$

$\lambda^2 + 9 = 0 \therefore \lambda = \pm 3i \Rightarrow \boxed{y_c = c_1\cos(3x) + c_2\sin(3x)}$

$y_p^n = A\cos(3x) + B\sin(3x) + C$, naive, it overlaps y_c.

$\boxed{y_p = x(A\cos(3x) + B\sin(3x)) + C}$

• Notice Cx will not work.

d.) $y'' + 8y' + 12y = e^{-2x} + 7x$

$\lambda^2 + 8\lambda + 12 = (\lambda+2)(\lambda+6) = 0 \Rightarrow \lambda_1 = -2,\ \lambda_2 = -6$

$\Rightarrow \boxed{y_c = c_1 e^{-2x} + c_2 e^{-6x}}$

$\boxed{y_p = Axe^{-2x} + Bx + C}$ has no overlap, it'll work.

e.) $y'' + 16y = xe^x\sin(x)$

$\lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i \Rightarrow \boxed{y = c_1\cos(4x) + c_2\sin(4x)}$

think about differentiating $xe^x\sin(x)$. The x can goto 1 but the e^x survives and $\sin(x)$ becomes $\cos(x)$ thus

$$\boxed{y_p = e^x(A\sin(x) + B\cos(x) + x(C\sin(x) + D\cos(x)))}$$

clearly no overlap here.

Bonus: We study howto solve $\frac{dy}{dx} - 3y = 13e^{\alpha x}$

So assume $Y_c = c_1 e^{\lambda x}$ lets notice what $Y_c' - 3y = 0$ means for λ. Ignore c_1 for now its unimportant,

$$\frac{dy_c}{dx} - 3y_c = \lambda e^{\lambda x} - 3e^{\lambda x} = e^{\lambda x}(\lambda - 3) = 0$$

$$\Rightarrow \lambda - 3 = 0 \quad \therefore \quad \underline{\lambda = 3}$$

Therefore the complementary sol^n is $\underline{Y_c = c_1 e^{3x}}$
We guess the Y_p now,

$$Y_p = Ae^{\alpha x}$$
$$Y_p' = \alpha Ae^{\alpha x}$$

$$Y_p' - 3y_p = \alpha Ae^{\alpha x} - 3Ae^{\alpha x} = 13e^{\alpha x}$$
$$A(\alpha - 3)e^{\alpha x} = 13e^{\alpha x}$$
$$A = \frac{13}{\alpha - 3} \quad \text{for } \alpha \neq 3$$

we could %.

Thus $\boxed{Y = c_1 e^{3x} + \frac{13}{\alpha - 3} e^{\alpha x} \qquad \alpha \neq 3}$

Now if $\alpha = 3$ we have overlap. Hence try

$$Y_p = Axe^{3x}$$
$$Y_p' = A(e^{3x} + 3xe^{3x})$$

$$Y_p' - 3Y_p = A(e^{3x} + 3xe^{3x}) - 3Axe^{3x} = 13e^{3x}$$
$$(A + \cancel{3xA} - \cancel{3Ax})e^{3x} = 13e^{3x}$$
$$\underline{A = 13}$$

$$\boxed{Y = c_1 e^{3x} + 13xe^{3x} \qquad \alpha = 3}$$

This is just to illustrate our methodology will work for any linear ordinary differential eq^n. For a n^{th} order ODE we'll get a n^{th} order charactersitic eq^n. <u>We studied $n=2$</u>

Sol^{n} to takehome part of test III

1. a.) $\frac{dY}{dx} = kY(1-\frac{1}{c}Y) = 0 \Rightarrow Y=0$ OR $1-\frac{Y}{c}=0$

Therefore the equilbrium sol^{n}'s are $\boxed{Y=0 \text{ and } Y=C}$

b.) Seperate variables, then integrate

$$(\star)\quad \int \frac{dY}{Y(1-\frac{1}{c}Y)} = \int k\,dx = kx + C_1$$

the rhs. was easy to integrate, the lhs. will require partial fractions

$$\frac{1}{Y(1-\frac{1}{c}Y)} = \frac{A}{Y} + \frac{B}{1-\frac{1}{c}Y}$$

$$1 = A(1-\tfrac{1}{c}Y) + BY$$

now use the roots to make life easy,

$Y=0 \mid \quad 1 = A$

$Y=C \mid \quad 1 = BC \Rightarrow B = 1/c$

Therefore,

$$\frac{1}{Y(1-\frac{1}{c}Y)} = \frac{1}{Y} + \frac{1/c}{1-\frac{1}{c}Y} = \frac{1}{Y} + \frac{1}{Y-C}$$

Thus,

$$\int \frac{dY}{Y(1-\frac{1}{c}Y)} = \int\left(\frac{1}{Y} + \frac{1}{Y-C}\right)dY$$

$$= \int \frac{1}{Y}dY + \int \frac{1}{Y-C}dY$$

$$= \ln|Y| - \ln|Y-C| + C_2$$

$$= \ln\left|\frac{Y}{Y-C}\right| + C_2$$

Now go back to the $(\star)$

$$\ln\left|\frac{Y}{Y-C}\right| = kx + C_3$$

$$\left|Y/(Y-C)\right| = e^{kx+C_3} = e^{C_3}e^{kx} = C_4 e^{kx}$$

$$\frac{Y}{Y-C} = \pm C_4 e^{kx} \Rightarrow Y = \pm C_4 e^{kx}(Y-C)$$

1b) Continued, $Y \mp C_4 e^{kx} Y = \mp C_4 e^{kx} c$

$$Y(1 \mp C_4 e^{kx}) = \mp C_4 e^{kx} \qquad \text{let } a \equiv \mp C_4$$

$$\boxed{Y = \frac{a e^{kx}}{1 + a e^{kx}}}$$

1c.) Orthogonal trajectories are sol^{n}'s to $\frac{dY}{dx} = \frac{-1}{kY(1-\frac{1}{c}Y)}$

$$\int k Y(1-\tfrac{1}{c}Y)\,dY = \int -dx = -x + C_1$$

$$\int (kY - \tfrac{k}{c}Y^2)\,dY = \tfrac{1}{2}kY^2 - \tfrac{1}{3c}kY^3 + C_2$$

$$\therefore \boxed{\tfrac{1}{2}kY^2 - \tfrac{1}{3c}kY^3 = -x + C_3}$$

aka $\frac{1}{2}Y^2 - \frac{1}{3}Y^3 = -kx + C_4$

1d.) let $k = c = 1$. Find sol^{n} and O.T. thru $(0, \frac{1}{2})$

sol^{n}: $\frac{1}{2} = \frac{a}{1+a} \Rightarrow \frac{1}{2} + \frac{1}{2}a = a \Rightarrow \frac{1}{2} = \frac{1}{2}a \Rightarrow \underline{a=1}$

O.T.: $\frac{1}{2}(\frac{1}{2})^2 - \frac{1}{3}(\frac{1}{2})^3 = C_3 = \frac{1}{4}(\frac{1}{2} - \frac{1}{6}) = \frac{1}{12} = C_3$

Thus $\boxed{Y = \frac{e^x}{1+e^x}}$ sol^{n} and $\boxed{\frac{1}{2}Y^2 - \frac{1}{3}Y^3 = -x + \frac{1}{12}}$ orthogonal trajectory

1e.) again $Y = \frac{ae^x}{1+ae^x}$ and $\frac{1}{2}Y^2 - \frac{1}{3}Y^3 = -x + C_3$

sol^{n}: $\frac{3}{2} = \frac{a}{1+a} \Rightarrow \frac{3}{2} + \frac{3}{2}a = a \Rightarrow \frac{3}{2} = -\frac{1}{2}a \Rightarrow \underline{a = -3}$

O.T. $\frac{1}{2}(\frac{3}{2})^2 - \frac{1}{3}(\frac{3}{2})^3 = C_3 = \frac{9}{8} - \frac{9}{8} = \underline{0 = C_3}$

$\boxed{Y = \frac{-3e^x}{1-3e^x} \text{ sol}^{n}}$ and $\boxed{\frac{1}{2}Y^2 - \frac{1}{3}Y^3 = -x \text{ O.T.}}$

(2) $y'' - 2y' + y = e^x + x\cos(2x) + x^4$

i.) $\lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0 \Rightarrow y_c = c_1 e^x + c_2 x e^x$

ii.) $y_p = \underbrace{Ax^2e^x}_{y_{p_1}} + \underbrace{x(B\cos(2x) + C\sin(2x)) + D\cos(2x) + E\sin(2x)}_{y_{p_2}} + \underbrace{Fx^4 + Gx^3 + Hx^2}_{y_{p_3}}$

$y_{p_1}' = A(2x + x^2)e^x$

$y_{p_1}'' = A(2 + 2x + 2x + x^2)e^x$

$$y_{p_1}'' - 2y_{p_1}' + y_{p_1} = A(x^2 + (2x + x^2)(-2) + 2 + 4x + x^2)e^x$$
$$= A(x^2 - 4x - 2x^2 + 2 + 4x + x^2)e^x$$
$$= A(2)e^x = e^x \Rightarrow A = 1/2$$

$$y_{p_2}' = B\cos(2x) + C\sin(2x) + x(-2B\sin(2x) + 2C\cos(2x)) - 2D\sin(2x) + 2E\cos(2x)$$

$$y_{p_2}' = \cos(2x)[B + 2Cx + 2E] + \sin(2x)[C - 2Bx - 2D]$$

$$y_{p_2}'' = -2\sin(2x)[B + 2Cx + 2E] + \cos(2x)[2C] + 2\cos(2x)[C - 2Bx - 2D] + \sin(2x)[-2B]$$

$$y_{p_2}'' = \cos(2x)[2C + 2C - 4Bx - 4D] + \sin(2x)[-2B - 4Cx - 4E - 2B]$$

$$x\cos(2x) = y_{p_2}'' - 2y_{p_2}' + y_{p_2} = \cos(2x)[4C - 4D - 4Bx] + \sin(2x)[-4E - 4B - 4Cx] - 2\cos(2x)[B + 2E + 2Cx] - 2\sin(2x)[C - 2D - 2Bx] + \cos(2x)[Bx + D] + \sin(2x)[Cx + E]$$

$\cos(2x)|$ $4C - 4D - 4Bx - 2B - 4E - 4Cx + Bx + D = x$

$x(-4B - 4C + B) + 4C - 3D - 2B - 4E = x$

$-3B - 4C = 1$ & $4C - 3D - 2B - 4E = 0$

$\sin(2x)|$ $-4E - 4B - 4Cx - 2C + 4D + 4Bx + Cx + E = 0$

$x(-4C + 4B + C) + (-4E + E - 4B - 2C + 4D) = 0$

$-3C + 4B = 0$ $\quad$ $-3E - 4B - 2C + 4D = 0$

(2) $\left.\begin{aligned} -3B - 4C &= 1 \\ 4C - 3D - 2B - 4E &= 0 \\ -3C + 4B &= 0 \\ -3E - 4B - 2C + 4D &= 0 \end{aligned}\right\} \sim \left[\begin{array}{cccc|c} -3 & -4 & 0 & 0 & 1 \\ -2 & 4 & -3 & -4 & 0 \\ 4 & -3 & 0 & 0 & 0 \\ -4 & -2 & 4 & -3 & 0 \end{array}\right]$ augmented coefficient matrix

enter the matrix above into TI and use "rref" to find,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3/25 \\ 0 & 1 & 0 & 0 & -4/25 \\ 0 & 0 & 1 & 0 & -22/125 \\ 0 & 0 & 0 & 1 & 4/125 \end{array}\right] \Longleftrightarrow \boxed{\begin{aligned} B &= -3/25 \\ C &= -4/25 \\ D &= -22/125 \\ E &= 4/125 \end{aligned}}$$

Next,

$$Y_{P_3} = Fx^4 + Gx^3 + Hx^2 + Ix + J$$
$$Y'_{P_3} = 4Fx^3 + 3Gx^2 + 2xH + I$$
$$Y''_{P_3} = 12Fx^2 + 6Gx + 2H$$

$$\begin{aligned} x^4 = Y''_{P_3} - 2Y'_{P_3} + Y_{P_3} &= 12Fx^2 + 6Gx + 2H \\ &\quad - 8Fx^3 - 6Gx^2 - 4Hx - 2I \\ &\quad + Fx^4 + Gx^3 + Hx^2 + Ix + J \\ &= 2H - 2I + J \\ &\quad + x(6G - 4H + I) \\ &\quad + x^2(12F - 6G + H) \\ &\quad + x^3(-8F + G) \\ &\quad + x^4(F) \end{aligned}$$

<u>Equating Coefficients</u>:

$F = 1$

$-8 + G = 0 \quad \therefore \; G = 8$

$12 - 48 + H = 0 \quad \therefore \; H = 36$

$48 - 144 + I = 0 \; \therefore \; I = 96$

$72 - 192 + J = 0 \; \therefore \; J = 120$

$$\boxed{\begin{aligned} Y = c_1 e^x + c_2 x e^x + \tfrac{1}{2}x^2 e^x + x\left(\tfrac{-3}{25}\cos(2x) - \tfrac{4}{25}\sin(2x)\right) - \tfrac{22}{125}\cos(2x) + \tfrac{4}{125}\sin(2x) + 2 \\ + x^4 + 8x^3 + 36x^2 + 96x + 120 \end{aligned}}$$

Clearly I was incorrect about how to set up x^4, needed I & J.

③ a.) $Y = A\sin(x+\varphi)$

$$= A\sin(x)\cos(\varphi) + A\cos(x)\cos(\varphi)$$
$$= (A\cos\varphi)\sin(x) + (A\cos\varphi)\cos(x)$$
$$= c_2\sin(x) + c_1\cos(x) \quad : \text{comparing} \Rightarrow \quad \boxed{\begin{aligned} C_1 &= A\cos\varphi \\ C_2 &= A\sin\varphi \end{aligned}}$$

b.) $Y = A\cosh(x) + B\sinh(x)$

$$= A\tfrac{1}{2}(e^{x}+e^{-x}) + B\tfrac{1}{2}(e^{x}-e^{-x})$$
$$= \tfrac{1}{2}(A+B)e^{x} + \tfrac{1}{2}(A-B)e^{-x}$$
$$= C_1e^{x} + C_2e^{-x} \quad \text{comparing} \Rightarrow \quad \boxed{\begin{aligned} C_1 &= \tfrac{1}{2}(A+B) \\ C_2 &= \tfrac{1}{2}(A-B) \end{aligned}}$$

c.) $Y'' - Y = \cosh(x)$

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \quad \therefore Y_c = C_1\cosh(x) + C_2\sinh(x)$

So there is overlap with $\cosh(x)$ multiply the naive guess by x,

$$Y_p = x(A\cosh(x) + B\sinh(x))$$
$$Y_p' = A\cosh(x) + B\sinh(x) + x(A\sinh(x) + B\cosh(x))$$
$$Y_p'' = A\sinh(x) + B\cosh(x) + A\sinh(x) + B\cosh(x) + Y_p$$

$$Y_p'' - Y_p = 2A\sinh(x) + 2B\cosh(x) + Y_p - Y_p = \cosh(x)$$

$$2A = 0 \rightarrow A = 0$$
$$2B = 1 \rightarrow B = 1/2$$

$$\boxed{Y = C_1\cosh(x) + C_2\sinh(x) + \frac{x}{2}\sinh(x)}$$