# MA 241-003 : Test IV. : Series & Sequences

1.) (10pts.) Prove that $a+ar+ar^2+ar^3+\cdots = \frac{a}{1-r}$ for $-1<r<1$.

2.) (10pts.) Prove that $0.9999\ldots = 1$, use the geometric series result from 1.

3.) (20pts.) Prove that the series below converge (or diverge). Explicitly and clearly explain the logic on which you base your claim.

| | |
|---|---|
| a.) | $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ |
| b.) | $\sum_{n=1}^{\infty} \left(\frac{n^2}{n^2+2}\right)$ |

4.) (30pts.) Find the power series centered at zero which represent the following functions. In each case give the interval of convergence (I.O.C.) for which the series converges.

| | |
|---|---|
| a.) | $\frac{x}{1-3x}$ |
| b.) | $\frac{1}{(1+x)^2}$ |
| c.) | $\tan^{-1}(x)$ |

5.) (20pts.) Use the known Maclaurin series to find the power series centered at zero that represent the following functions,

| | |
|---|---|
| a.) | $\sin(2x)$ |
| b.) | $xe^{x^2}$ |

6.) (10pts.) Prove that the power series you found in #5a converges for all values of x. Do this by applying the ratio test.

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① See notes.

② $0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots$

Clearly geometric with $a = 9/10$ and $r = 1/10$ thus $0.999\ldots = \frac{9/10}{1-1/10} = \frac{9/10}{9/10} = 1.$

③ a.) $s = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ is an alternating series with $b_n = \frac{1}{n}$.

Notice $b_n = \frac{1}{n} > 0$, $\frac{1}{n+1} < \frac{1}{n} \Rightarrow b_{n+1} < b_n$ and $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

Therefore s converges by A.S.T.

b.) Notice $\lim_{n\to\infty}\left(\frac{n^2}{n^2+2}\right) = \lim_{n\to\infty}\left(\frac{1}{1+\underbrace{2/n^2}_{0}}\right) = 1 \neq 0 \therefore \sum_{n=1}^{\infty}\frac{n^2}{n^2+2}$ diverges by $n^{th}$ term test.

④ a.) $\frac{x}{1-3x} = \frac{a}{1-r}$ with $a = x$, $r = 3x$ when $|3x| < 1 \Rightarrow |x| < 1/3$,
$\Rightarrow$ I.O.C. $= (-1/3, 1/3)$

Hence $\frac{x}{1-3x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} x(3x)^n = \boxed{\sum_{n=0}^{\infty} 3^n x^{n+1} = \frac{x}{1-3x}}$

(Which has the 1st three terms, $\frac{x}{1-3x} = x + 3x^2 + 9x^3 + \cdots$)

b.) $\int \frac{dx}{(1+x)^2} = \frac{-1}{1+x} + C$ geometric with $a = -1$ and $r = -x$ applies when $|r| < 1 \Rightarrow |-x| < 1 \Rightarrow \boxed{-1 < x < 1}$ I.O.C.

$= \sum_{n=0}^{\infty}(-1)(-x)^n + C$

Now notice $\frac{d}{dx}\int\frac{dx}{(1+x)^2} = \frac{1}{(1+x^2)^2}$ but we can also diff. the series term by term,

$$\frac{1}{(1+x^2)^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)(-x)^n + C\right)$$

$$= \sum_{n=0}^{\infty}(-1)n(-x)^{n-1}\frac{d}{dx}(-x)$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^{n-1}n\,x^{n-1} = 1 - 2x + 3x^2 \cdots \text{ for } -1 < x < 1}$$

(4)(c) $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ using $a=1$, $r=-x^2$

Again the I.O.C. follows from geom. series result

$|r|<1 \Rightarrow |-x^2|<1 \Rightarrow |x^2|<1 \Rightarrow \boxed{-1<x<1 \quad \text{I.O.C}}$

Now integrate to get back to $\tan^{-1}(x)$,

$$\tan^{-1}(x) = \int \frac{d}{dx}\left(\tan^{-1}(x)\right)dx$$

$$= \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,dx + \tilde{C}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C$$

note $\tan^{-1}(0) = 0 = C \therefore \underline{C=0}$.

Thus $\boxed{\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots}$

(5.) a.) $\sin(2x) = \sin(u) \quad : \quad u = 2x$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!} = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \cdots}$$

b.) $xe^{x^2} = xe^u \quad : \quad u = x^2$

$$= x\sum_{n=0}^{\infty}\frac{u^n}{n!}$$

$$= \sum_{n=0}^{\infty} x\frac{(x^2)^n}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!} = x + x^2 + \frac{1}{2}x^3 + \cdots = xe^{x^2}}$$

6.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad : \text{consider ratio test for } \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$= \lim_{n\to\infty} \left| \frac{2^{2(n+1)+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{2^{2n+1} x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2^{2n+3} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{2n+1} x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| 2^2 x^2 \frac{(2n+1)!}{(2n+3)!} \right|$$

$$= 4x^2 \lim_{n\to\infty} \left( \frac{\cancel{(2n+1)}!}{(2n+3)(2n+2)\cancel{(2n+1)}!} \right)$$

$$= 4x^2 \lim_{n\to\infty} \left( \frac{1}{(2n+3)(2n+2)} \right) \to 0$$

$= 0 < 1$ ∴ The series converges independent of $x$ by the ratio test.