Quiz on U-substitution and Int. by Parts.

① $\int \sin^3(2x+1)\,dx = \int \sin^3(w)\,\frac{dw}{2}$ ← $\boxed{\begin{array}{l} w = 2x+1 \\ dw = 2dx \;\therefore\; dx = \frac{1}{2}dw \end{array}}$

$= \frac{1}{2}\int \sin^2(w)\sin(w)\,dw$

$= \frac{1}{2}\int (1-\cos^2(w))\sin(w)\,dw$

$= \frac{1}{2}\int (1-u^2)(-du)$ ← $\boxed{\begin{array}{l} u = \cos(w) \\ du = -\sin(w)\,dw \end{array}}$

$= \frac{1}{2}\int (u^2-1)\,du$

$= \frac{1}{2}\left(\frac{u^3}{3} - u\right) + C$

$= \frac{1}{2}\left(\frac{1}{3}\cos^3(2x+1) - \cos(2x+1)\right)$

② $\int 4x^3 e^{x^4}\,dx = \int e^u\,du = e^u + C = \boxed{e^{x^4} + C}$ letting $u = x^4$, $du = 4x^3dx$

③ $\int \frac{dx}{25x^2+16} = \int \frac{dx}{16\left(\frac{25x^2}{16}+1\right)}$

$= \int \frac{dx}{16((5x/4)^2+1)}$

$= \frac{4}{5}\int \frac{du}{u^2+1}\cdot\frac{1}{16}$ ← $\boxed{u = \frac{5x}{4} \Rightarrow dx = \frac{4}{5}du}$

$= \frac{1}{16}\frac{4}{5}\tan^{-1}(u) + C$

$= \frac{1}{16}\frac{4}{5}\tan^{-1}\left(\frac{5x}{4}\right) + C$

④ $\int \underbrace{x^2}_{u}\underbrace{\sin(x)\,dx}_{dv} = -x^2\cos(x) + \int \overbrace{2x}^{u}\,\overbrace{\cos(x)dx}^{dv}$

$= -x^2\cos(x) + \left[2x\sin(x) - \int 2\sin(x)\,dx\right]$

$= \boxed{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}$

QUIZ IV : PARTIAL FRACTIONS & TRIG. SUBSTITUTION

① Set up partial-fractal decomposition of $\dfrac{3x^2-5x+3}{(x^2+1)(x^2+2x+1)(x^2+2)^2(x-3)}$

Answer: no long % needed bc deg(num) < deg(denominator). Also notice $x^2+2x+1 = (x+1)^2$ thus

$$\frac{3x^2-5x+3}{(x^2+1)(x+1)^2(x^2+2)^2(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{(x^2+2)} + \frac{Gx+H}{(x^2+2)^2} + \frac{I}{x-3}$$

② $\int \csc(x)\,dx = \int \frac{1}{\sin(x)}dx$

$= \int \frac{\sin(x)}{\sin^2(x)}dx$

$= \int \frac{\sin(x)\,dx}{1-\cos^2(x)}$

$= \int \frac{-du}{1-u^2}$ ← $\boxed{u = \cos(x),\ du = -\sin(x)dx}$

$= \frac{-1}{2}\int\frac{du}{u+1} + \frac{1}{2}\int\frac{du}{u-1}$ $\qquad \frac{-1}{1-u^2} = \frac{1}{u^2-1} = \frac{1}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1} = \frac{-1/2}{u+1} + \frac{1/2}{u-1}$

Some work.

$= \frac{-1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1| + C$

$= \boxed{\frac{-1}{2}\ln|\cos(x)+1| + \frac{1}{2}\ln|\cos(x)-1| + C}$

③ $\int\sqrt{a^2-x^2}\,dx = \int(a\cos\theta)(a\cos\theta\,d\theta)$ $\qquad x = a\sin\theta$, $\sqrt{a^2-x^2} = \sqrt{a^2\cos^2\theta} = a\cos\theta$, $dx = a\cos\theta\,d\theta$

$= \int a^2\cos^2\theta\,d\theta$

$= \int\frac{a^2}{2}(1+\cos 2\theta)\,d\theta$

$= \boxed{\frac{a^2}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C}$

④ $\int \tan^2\theta\,d\theta = \int(\sec^2\theta - 1)\,d\theta$ ← $\boxed{\tan^2\theta + 1 = \sec^2\theta \quad \therefore \tan^2\theta = \sec^2\theta - 1}$

$= \boxed{\tan\theta - \theta + C}$

Quiz IV

① This one has both types of impropreity,

$$\int_0^\infty \frac{1}{x^2}\,dx = \int_0^1 \frac{1}{x^2}\,dx + \int_1^\infty \frac{1}{x^2}\,dx$$

Break it up into pieces,

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$$ ← def^n of integrating near V.A. of fnct.

$$= \lim_{t\to 0^+}\left(\frac{-1}{x}\Big|_t^1\right)$$

$$= \lim_{t\to 0^+}\left(\frac{-1}{1} + \underbrace{\frac{1}{t}}_{\to\infty}\right)$$

$= \infty$. (We could stop here and say $\int_0^\infty \frac{1}{x^2}dx$ diverges.)

I'll do the other ½ just to see how it goes,

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(\int_1^t \frac{1}{x^2}\,dx\right)$$ ← def^n of integrating to ∞.

$$= \lim_{t\to\infty}\left(\frac{-1}{x}\Big|_1^t\right)$$

$$= \lim_{t\to\infty}\left(\underbrace{\frac{-1}{t}}_{\to 0} + \frac{1}{1}\right)$$

$$= 1$$

Thus to collect our results,

$$\boxed{\int_0^\infty \frac{1}{x^2}\,dx = \infty \text{, that is it diverges}}$$

② Show that the area of circle is πr^2 is in notes on pg. 46.
where $r = a$

Quiz 5

① $\frac{dy}{dx} = e^{x+y}$ solve by seperation of variables

Crucial algebraic step : $e^{x+y} = e^x e^y$

$$\frac{dy}{dx} = e^x e^y \Rightarrow e^{-y} dy = e^x dx$$

Integrate both sides to get:

$$-e^{-y} = e^x + k$$

$$e^{-y} = c - e^x \qquad (c = -k)$$

$$-y = \ln(c - e^x)$$

$$y = -\ln(c - e^x)$$

$$\therefore \boxed{y = \ln\left(\frac{1}{c - e^x}\right)}$$

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② Show that $y = \sin(x)$ is a sol^{n} to $\left(\frac{dy}{dx}\right)^2 + y^2 = 1$

Well notice $\frac{dy}{dx} = \cos(x)$ then substitute

$$\left(\frac{dy}{dx}\right)^2 + y^2 = \left(\cos(x)\right)^2 + \left(\sin(x)\right)^2 = 1$$

Thus $y = \sin(x)$ is a sol^{n} to the diff. eq^{n}.

Quiz 6 : Homogeneous DEq's

① $Y'' = 0$ given $Y(0) = 0$ and $Y(1) = 1$ find the sol^n.

$\lambda^2 = 0 \Rightarrow \lambda = \pm 0 \Rightarrow Y = c_1 e^0 + c_2 x e^0 = c_1 + c_2 x = Y$ gen. sol^n.

Apply boundary conditions:

$Y(0) = c_1 + c_2(0) = 0 \Rightarrow c_1 = 0$

$Y(1) = c_1 + c_2 = 1 \Rightarrow c_2 = 1 \quad \therefore \boxed{Y = X}$

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② $Y'' - 2Y' + 2Y = 0$ find general sol^n.

$\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i \quad \therefore \alpha = 1$ & $\beta = 1$

Hence $\boxed{Y = e^x [c_1 \cos(x) + c_2 \sin(x)]}$

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③ $y'' + 5y' + 6y = 0$ given $Y(0) = 0$ and $Y'(0) = 1$ find sol^n.

$\lambda^2 + 5\lambda + 6 = 0 \Rightarrow (\lambda+3)(\lambda+2) = 0 \therefore \lambda = -3$ or -2

Hence $Y = c_1 e^{-3x} + c_2 e^{-2x}$ is gen. sol^n.

$Y' = -3c_1 e^{-3x} - 2c_2 e^{-2x}$

Now apply intal conditions,

$Y(0) = c_1 + c_2 = 0 \quad \therefore -c_1 = +c_2$

$Y'(0) = -3c_1 - 2c_2 = \boxed{c_2 = 1} \quad \therefore \boxed{c_1 = -1}$

Thus $\boxed{Y = -e^{-3x} + e^{-2x}}$

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④ $Y'' + 25Y = 0$ given $Y(0) = 1$ and $Y'(0) = 0$

$\lambda^2 + 25 = 0 \Rightarrow \lambda = \pm 5i \quad \therefore Y = c_1 \cos(5x) + c_2 \sin(5x)$

$Y' = -5c_1 \sin(5x) + 5c_2 \cos(5x)$

Apply intial conditions

$Y(0) = c_1 = 1$

$Y'(0) = -5c_2 = 0 \quad \therefore \boxed{Y = \cos(5x)}$

QUIZ VII

① $\boxed{y'' + 6y' + 5y = 5t + 35 \text{ find general sol}^{\underline{n}}\text{, then find specific sol}^{\underline{n}} \text{ that has } y(0) = 0 \text{ and } y'(0) = 1}$

$\lambda^2 + 6\lambda + 5 = 0 \Rightarrow (\lambda+5)(\lambda+1) = 0 \quad \therefore \lambda = -5 \text{ or } \lambda = -1$

Thus the homogeneous sol$^{\underline{n}}$ is $\boxed{y_H = c_1 e^{-5t} + c_2 e^{-t}}$

There is no over lap with t and y_H so we make the usual guess for y_p when we have t,

$$y_p = At + B$$
$$y_p' = A$$
$$y_p'' = 0$$

Substituting,

$$y_p'' + 6y_p' + 5y_p = 5t + 35$$
$$6(A) + 5(At + B) = 5t + 35$$
$$(6A + 5B) + t(5A) = 35 + 5t$$

$6A + 5B = 35$

$5A = 5$

(from equating coefficients)

Do some algebra to see $\boxed{A = 1}$ and $\boxed{B = 1}$. Thus $\boxed{y_p = t + 1}$

Hence $\boxed{y_{gen} = c_1 e^{-5t} + c_2 e^{-t} + t + 1}$

$$y_{gen}' = -5c_1 e^{-5t} - c_2 e^{-t} + 1$$

Apply intial conditions to find specific sol$^{\underline{n}}$,

$$y(0) = 0 = c_1 + c_2 + 1$$
$$y'(0) = 1 = -5c_1 - c_2 + 1 \Rightarrow c_2 = -5c_1 \Rightarrow -4c_1 + 1 = 0 \quad \therefore c_1 = \tfrac{1}{4} \quad \therefore c_2 = \tfrac{-5}{4}$$

Thus $\boxed{y(t) = \frac{1}{4}e^{-5t} - \frac{5}{4}e^{-t} + t + 1}$ (as a check note $y(0) = 0$)

② For each of the follwing find the homogeneous sol$^{\underline{n}}$ and state a correct guess for y_p (But don't determine the coefficients). (I omit some details in this sol$^{\underline{n}}$)

(a.) $y'' + 9y = \sin(x) \Rightarrow y_H = c_1\cos(3x) + c_2\sin(3x)$

$$y_p = A\sin(x) + B\cos(x)$$

(b.) $y'' + 6y' + 5y = e^x\sin(x) \Rightarrow y_H = c_1 e^{-5t} + c_2 e^{-t}$

$$y_p = e^x(A\sin(x) + B\cos(x))$$

(c.) $y'' + 2y' + y = xe^{-x} \Rightarrow y_H = c_1e^{-x} + c_2xe^{-x}$

$$y_p = x^2(Axe^{-x} + Be^{-x})$$

double overlap of y_H with usual guess for y_p.

(d.) $y'' + 4y' + \frac{25}{4}y = x^2 \Rightarrow y_H = e^{-2x}\left(c_1\cos(\tfrac{3}{2}x) + c_2\sin(\tfrac{3}{2}x)\right)$

$$y_p = Ax^2 + Bx + C.$$

$$\left(\begin{array}{l} \lambda^2 + 4\lambda + \frac{25}{4} = 0 \\ \lambda = \frac{-4 \pm \sqrt{16 - 25}}{2} = -2 \pm \frac{3i}{2} \end{array}\right)$$

QUIZ 8 : SEQUENCES

Find the limit of each sequence below if it converges, otherwise state that the limit diverges.

① $\lim_{n\to\infty} \left(\frac{1}{5^n}\right) = 0$ (P.L.A. $\equiv$ "principal of least astonishment")

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② $\lim_{n\to\infty} \left(\frac{n^2-1}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1-\overbrace{1/n^2}^{0}}{1+\underbrace{1/n^2}_{0}}\right) = 1$ (divided numerator & denominator by n^2)

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③ $\lim_{n\to\infty}\left(\frac{4n-3}{3n+4}\right) = \lim_{n\to\infty}\left(\frac{4-\overbrace{3/n}^{0}}{3+\underbrace{4/n}_{0}}\right) = \frac{4}{3}$ (divided num. & denom by n.)

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④ $\lim_{n\to\infty}\left(\frac{n-\sqrt{n}}{n+2}\right) = \lim_{n\to\infty}\left(\frac{-\sqrt{n}}{1+2/n}\right) = \infty$ (nothing can kill the $\sqrt{n}$ up-top, it drives the limit to ∞)

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⑤ $\lim_{n\to\infty}\left(\sqrt{n+2}-\sqrt{n}\right) = \lim_{n\to\infty}\left(\left(\sqrt{n+2}-\sqrt{n}\right)\left[\frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}}\right]\right)$

$= \lim_{n\to\infty}\left(\frac{n+2-n}{\sqrt{n+2}+\sqrt{n}}\right)$

$= \lim_{n\to\infty}\left(\frac{2}{\sqrt{n+2}+\sqrt{n}}\right)$

$= 0$

By the way you could <u>not</u> do this limit by breaking it up

$\underbrace{\lim_{n\to\infty}\sqrt{n+2} - \lim_{n\to\infty}\sqrt{n} = \infty - \infty = 0}$

bogus argument!

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⑥ $\lim_{n\to\infty}(0) = 0$ notice $0 = 2n - 2n$ { WARNING THESE CALCULATIONS SHOW YOU WHAT <u>NOT</u> TO DO!

$\lim_{n\to\infty}(2n-2n) = \lim_{n\to\infty}(2n) - 2\lim_{n\to\infty}(n) = \infty - 2\infty = -\infty$

$\therefore \lim_{n\to\infty}(0) = \boxed{-\infty = 0}$!!!

this is the consequence of assuming you can do arithmatic on ∞. Bottom line you must be careful with ∞.

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⑦ $\lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) \underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{1}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

where we extended n to be a continous variable to apply L'Hopital's Rule.

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⑧ $\lim_{n\to\infty}\left(n\,2^{-n}\right) = \lim_{n\to\infty}\left(\frac{n}{2^n}\right) \underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{n\to\infty}\left(\frac{1}{\ln(2)2^n}\right) = 0$

Quiz 9 : Series

① $\boxed{\text{Calculate the exact fraction that} = 0.131313\ldots = 0.\overline{13}}$

$0.\overline{13} = \frac{13}{100} + \frac{13}{100^2} + \frac{13}{100^3} + \cdots \Rightarrow a = \frac{13}{100}$ & $r = \frac{1}{100}$, its a geometric series

Thus $0.\overline{13} = \frac{a}{1-r} = \frac{13/100}{1 - 1/100} = \frac{13}{100-1} = \boxed{\frac{13}{99}}$ (corrected from original.)

② $\boxed{\text{Does } \sum_{n=1}^{\infty} \frac{1}{n^2+3} \text{ diverge or converge and why?}}$

$\frac{1}{n^2+3} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by $p=2$ series test $\overset{}{\underset{D.C.T.}{\Longrightarrow}}$ $\boxed{\sum \frac{1}{n^2+3} \text{ converges}}$

③ $\boxed{\text{Does } \sum_{n=1}^{\infty} \frac{1}{n!} \text{ diverge or converge and why?}}$ use ratio test,

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{n!}{(n+1)!} \right) = \lim_{n\to\infty} \left| \frac{n!}{(n+1)n!} \right| = \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0 < 1$

Thus by ratio test $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

④ $\boxed{\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) \text{ converges find it's sum.}}$

See notes pg. 102 $\boxed{E6}$

⑤ $\boxed{\sum_{n=1}^{\infty} \tan^{-1}(n) \text{ diverge or converge?}}$ It diverges by n^{th} term test $\tan^{-1}(n) \longrightarrow \frac{\pi}{2}$ as $n \to \infty$.

⑥ $\boxed{\sum \frac{(-1)^n}{n} \text{ diverge or converge?}}$ converges by alternating series test; $b_n = \frac{1}{n}$ where $\frac{1}{n+1} < \frac{1}{n} \Rightarrow b_{n+1} < b_n$ & $b_n > 0$ and $b_n \longrightarrow 0$ as $n \to \infty$ ∴ Alternating Series Test yields $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv.

⑦ $S(x) = \sum_{n=0}^{\infty} \left(\frac{3x}{7} + 1 \right)^n = 1 + \left(\frac{3x}{7} + 1 \right) + \left(\frac{3x}{7} + 1 \right)^2 + \cdots$ $a = 1$, $r = \frac{3x}{7} + 1$ geom. series

Converges to $\frac{a}{1-r} = \frac{1}{1 - \frac{3x}{7} - 1} = \frac{-7}{3x}$ for $|r| < 1$ ∴

That means $\left| \frac{3x}{7} + 1 \right| < 1 \Rightarrow \left| x + \frac{7}{3} \right| < \frac{7}{3}$ ∴ $\boxed{R = 7/3}$ $\boxed{(-\frac{14}{3}, 0)}$

$I.O.C = \left(-\frac{7}{3} - \frac{7}{3}, -\frac{7}{3} + \frac{7}{3} \right)$