MA241-006: Calculus II

Instructor: Mr. James Cook
Test: #4
Date: Friday, April 27, 2006

Directions: You **must** show **ALL** your work to receive credit. This means you should should explain why you gave the answer that you did, on this test you should explain what test or result you base your thoughts on.

1. (16 pts) Consider the geometric series $s = a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$

a.) $S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{\cdots} + \cancel{ar^n}$. Calculate a nice formula for this n-th partial sum.

$-rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots \cancel{+} ar^{n+1}$

$S_n - rS_n = a - ar^{n+1}$

$S_n(1-r) = a - ar^{n+1} \Rightarrow S_n = \frac{a}{1-r} - \frac{ar^{n+1}}{1-r}$

b.) If $|r| < 1$ prove that $s = \frac{a}{1-r}$.

$$s = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^{n+1}}{1-r}\right)$$

$$= \lim_{n\to\infty}\left(\frac{a}{1-r}\right) - \frac{a}{1-r}\cancel{\lim_{n\to\infty}(r^{n+1})}^{\to 0} \quad \text{if } |r| < 1$$

$$= \boxed{\frac{a}{1-r}}$$

c.) If $|r| \geq 1$ prove that s diverges. (use the appropriate test)

$\sum_{n=1}^{\infty} ar^{n-1}$ has typical term $a_n = ar^{n-1}$

notice $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(ar^{n-1}) \neq 0$ if $|r| \geq 1$

$\therefore$ the geometric series diverges for $|r| \geq 1$.

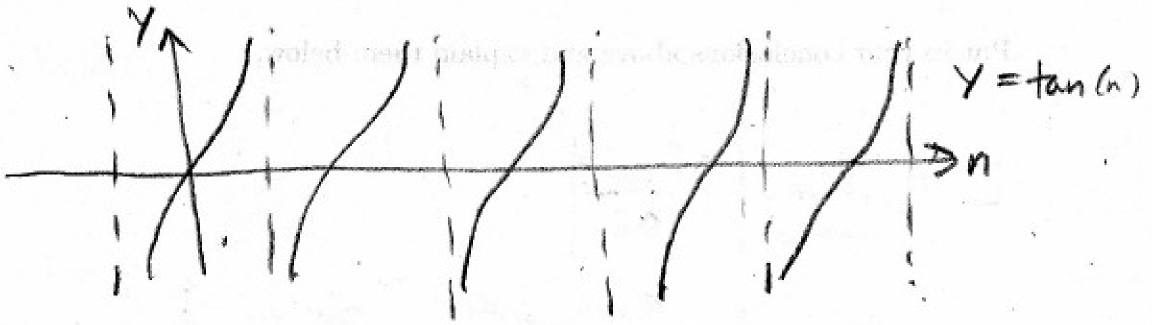
d.) Calculate the sum $1 + 1/2 + (1/2)^2 + \ldots$.

$1 + 1/2 + (1/2)^2 + \cdots = \frac{a}{1-r} = \frac{1}{1-1/2} = \frac{1}{1/2} = \boxed{2}$

2. (10 pts) Determine wether the following series converges or diverges. Explain your reasoning and make sure to verify any assumptions that the test you use requires.

$$\sum_{n=1}^{\infty} tan(n)$$

Observe that tan(n) is periodic with period π. It's graph looks something like



Clearly $\lim_{n\to\infty}(\tan(n))$ d.n.e $\therefore \lim_{n\to\infty}(\tan(n)) \neq 0$

Hence, by n^{th} term test the series diverges.

3. (14 pts) Use the ratio test to determine the largest open interval of convergence (this means you do not need to worry about the endpoints) for the power series below. Also determine where the series is centered and what the radius of convergence is.

Sorry about that I should've put in some spaces.

$$\sum_{n=1}^{\infty} \frac{1}{3n} 4^n (x+5)^n$$

Open I.O.C = (-5.25, -4.75)
Radius of convergence = R = 1/4
center = a = -5

Put in your conclusions above and explain them below,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{4^{\cancel{n+1}}(x+5)^{n+1}}{\cancel{3}(n+1)} \frac{\cancel{3}n}{4^n \cancel{(x+5)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n}{n+1} 4(x+5) \right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right) 4|x+5|$$

$$= 4|x+5| \cancel{\lim_{n\to\infty} \left(\frac{n}{n+1}\right)} \to 1.$$

Thus $L = 4|x+5|$ then by ratio test we can conclude the series will converge for those x which cause $L < 1$, that is,

$$4|x+5| < 1 \Rightarrow -1 < 4(x+5) < 1$$

$$\Rightarrow -1/4 < x+5 < 1/4$$

$$\Rightarrow -\frac{1}{4} - 5 < x < -5 + 1/4$$

$$\Rightarrow -5.25 < x < -4.75$$

$\Rightarrow$ open I.O.C. = $(-5.25, -4.75)$ with $R = 1/4$ and center = -5.

4. (25 pts) Use the geometric series result to find the complete power series representation for the functions below (use "sigma" notation). You may need to integrate and/or differentiate to make it work (we referred to that as the "geometric series trick"). For 2/3 partial credit you may find just the first three terms.

a.) $f(x) = ln(1 + x^2)$

used $a = 2x$ and $r = -x^2$

$$f'(x) = \frac{2x}{1+x^2} = \sum_{n=0}^{\infty} 2x(-x^2)^n = \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1}$$

$$f(x) = \int f'(x)dx = \int \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1} dx$$

$$= \sum_{n=0}^{\infty} 2(-1)^n \int x^{2n+1} dx$$

$$= \sum_{n=0}^{\infty} \cancel{2}(-1)^n \frac{x^{2n+2}}{\cancel{2}n+\cancel{2}} + C = \ln(1+x^2)$$

But $C = 0$ because $f(0) = C = \ln(1) = 0$. Thus

$$\boxed{\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}}$$

Remark: ignoring my directions, one could also use known Maclaurin series for $\ln(1+u)$ with $u = x^2$. That is however not what I asked for here.

b.) $g(x) = \frac{3}{1-x^2}$

$$g(x) = \frac{3}{1-x^2} = \sum_{n=0}^{\infty} 3(x^2)^n = \boxed{\sum_{n=0}^{\infty} 3x^{2n} = \frac{3}{1-x^2}}$$

5. (10 pts) Use the known Maclaurin series to calculate the power series expansions about zero for $f(x) = x cos(x^3)$. You may find just the first three non-zero terms for this problem,

$$f(x) = x\cos(x^3)$$

$$= x\cos(u) \qquad u = x^3$$

$$= x\left(1 - \frac{1}{2}u^2 + \frac{1}{4!}u^4 + \cdots\right)$$

$$= x\left(1 - \frac{1}{2}(x^3)^2 + \frac{1}{24}(x^3)^4 + \cdots\right)$$

$$= \boxed{x - \frac{1}{2}x^7 + \frac{1}{24}x^{13} + \cdots}$$

6. (15 pts) Use Taylor's theorem to find the first three non-zero terms in the power series expansion centered at $a = 2$ for the function $f(x) = x^3$.

$$f(x) = f(2) + f'(2)(x-2) + \frac{1}{2} f''(2)(x-2)^2 + \cdots$$

So we need to calculate,

$$f(2) = 2^3 = 8$$

$$f'(2) = (3x^2|_{x=2} = 12$$

$$f''(2) = (6x|_{x=2} = 12$$

Thus,

$$f(x) = 8 + 12(x-2) + \frac{1}{2}(12)(x-2)^2 + \cdots$$

$$\boxed{x^3 = 8 + 12(x-2) + 6(x-2)^2 + \cdots}$$

I suppose it'd have been more interesting if I'd asked for the whole Taylor series, its only one more term,

$$f'''(2) = 6 \quad \text{and} \quad f^{(n)}(2) = 0 \quad \text{for } n \geq 4.$$

Giving, (since $\frac{6}{3!} = 1$),

$$\boxed{x^3 = 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3}$$

In other words the "$+\cdots$" in the answer is just one term.

7. (15 pts) Find the power series solution to the integral below (use the result from 4a). For full credit your answer should be given in the "sigma" notation, again you may obtain 2/3 credit for the first three non-zero terms.

$$\int ln(1+x^2)\; dx$$

$$\int \ln(1+x^2)\,dx = \int\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}\right) dx \quad \text{(using 4a.)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \int x^{2n+2}\,dx$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n+1}\right)\left(\frac{1}{2n+3}\right) x^{2n+3} + C}$$