YOUR NAME HERE:

Instructor: James Cook , MA 241-003 Calculus II, Test I: Integration, July 9, 2007

Directions: Show your work, if you doubt that you've shown enough detail then ask. No electronic aids of any sort are permitted modulo your watch and a scientific calculator.

1. (40 pts) Integrate the following,

   (a.) $\int x\cos(x^2+3)dx$ (c.) $\int_0^1 (4x+3)^{97}dx$ (b.) $\int e^x\cos(x)dx$ (d.) $\int \sin^3(\theta)d\theta$

2. (20 pts) Integrate the following using partial fractions,

$$\int \frac{x^2+7}{x^3+4x}\,dx$$

3. (5pts) Use the substitution $u = \sec(\theta)+\tan(\theta)$ to calculate the following integral,

$$\int \sec(\theta)d\theta$$

4. (15pts) Use a trigonometric substitution to complete the following integral. For full credit leave your answer in terms of algebraic functions as much is possible.

$$\int \frac{1}{\sqrt{x^2-25}}\,dx$$

5. (10pts) Set up the partial fractions decomposition for the rational function below, do not explicitly calculate the $A, B, C, \dots$.

$$f(x) = \frac{x+23}{(x^2-4)^2(x+3)(x^2+9)^2}$$

6. (10pts) Calculate the following improper integrals, 3pts for setting up limits, 2pts for integrating and evaluating the limits, don't panick if you can't take the limit, all is not lost.

$$(a.)\ \int_0^\infty \frac{1}{1+x^2}dx \qquad (b.)\ \int_0^1 \frac{1}{x}dx$$

   for a bonus point sketch the areas calculated by the integrals above.

7. BONUS:(2pts) Calculate the following,
   (3pts): $\int \csc(\theta)d\theta$.
   (2pts): $\int \frac{x+5}{(x^2+4x+5)^2}\,dx$.
   (20pts): $\int \frac{1}{1+\sin(x)+\cos(x)}dx$,
   WARNING: don't even try this last one unless you have everything else completely finished and checked, this one is nontrivial.

Ma 241 - 003, summer II 2007, TEST ONE

1.(a.) $\int x\cos(x^2+3)\,dx = \int x\cos(u)\,\frac{du}{2x}$ $\leftarrow$ $u = x^2+3$, $du = 2x\,dx$

$= \frac{1}{2}\int \cos(u)\,du$

$= \frac{1}{2}\sin(u) + C$

$= \boxed{\frac{1}{2}\sin(x^2+3) + C}$

(b.) $\int_0^1 (4x+3)^{97}\,dx = \int_3^7 u^{97}\,\frac{du}{4}$ $\leftarrow$ $u = 4x+3$, $du = 4dx$, $dx = \frac{1}{4}du$, $u(0) = 4\cdot 0+3 = 3$, $u(1) = 4\cdot 1+3 = 7$

$= \frac{1}{4}\cdot\frac{1}{98}\,u^{98}\Big|_3^7$

$= \boxed{\frac{1}{392}\left(7^{98} - 3^{98}\right)}$

(c.) $I = \int e^x\cos(x)\,dx = uv - \int v\,du$ — $u = e^x$, $dv = \cos(x)dx$, $du = e^x dx$, $v = \sin(x)$

$= e^x\sin(x) - \int e^x\sin(x)\,dx$

$= e^x\sin(x) - uv + \int v\,du$ — $u = e^x$, $dv = \sin(x)dx$, $du = e^x dx$, $v = -\cos(x)$

$= e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx$

$\Rightarrow 2I = e^x(\cos(x) + \sin(x))$ $\Rightarrow$ $\boxed{\int e^x\cos(x)\,dx = \frac{1}{2}e^x(\cos(x)+\sin(x)) + C}$

(d.) $\int \sin^3\theta\,d\theta = \int (1-\cos^2\theta)\sin\theta\,d\theta$ — $u = \cos\theta$, $du = -\sin\theta\,d\theta$, $-du = \sin\theta\,d\theta$

$= \int (1-u^2)(-du)$

$= \int (u^2-1)\,du$

$= \frac{u^3}{3} - u + C$

$= \boxed{\frac{\cos^3\theta}{3} - \cos\theta + C}$

2.) $$\frac{x^2+7}{x^3+4x} = \frac{x^2+7}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$x^2+7 = A(x^2+4) + (Bx+C)x$$
$$= (A+B)x^2 + Cx + 4A$$

Equating Coefficients yields

$$\begin{aligned} A+B &= 1 \\ C &= 0 \\ 4A &= 7 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= 7/4 \\ B &= 1-A = -3/4 \\ C &= 0 \end{aligned}$$

← ha! you guys got lucky. Not my intention.

Hence,

$$\int \frac{x^2+7}{x^3+4x}\,dx = \int \frac{7/4}{x}\,dx - \int \frac{3}{4}\frac{x}{x^2+4}\,dx$$

$u = x^2+4$
$du = 2x\,dx$
$x\,dx = \frac{du}{2}$

$$= \frac{7}{4}\ln|x| - \frac{3}{8}\int \frac{du}{u}$$

$$= \boxed{\frac{7}{4}\ln|x| - \frac{3}{8}\ln|x^2+4| + C}$$

3.) If $u = \sec\theta + \tan\theta$

$$du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta = \sec\theta\,\underbrace{(\tan\theta + \sec\theta)}_{u}\,d\theta$$

thus $du/u = \sec\theta\,d\theta$.

$$\int \sec\theta\,d\theta = \int \frac{du}{u} = \ln|u| + C$$

$$= \boxed{\ln|\sec\theta + \tan\theta| + C}$$

4.) $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx = \int \frac{5\sec\theta\tan\theta\,d\theta}{\sqrt{25\tan^2\theta}}$

$\displaystyle = \int \frac{5\sec\theta\tan\theta\,d\theta}{5\tan\theta}$

$\displaystyle = \int \sec\theta\,d\theta$

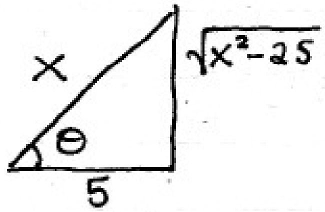
$= \ln|\sec\theta + \tan\theta| + C$

$\displaystyle = \boxed{\ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + C}$

$x = 5\sec\theta$

$x^2 - 25 = 25\sec^2\theta - 25$
$= 25(\sec^2\theta - 1)$
$= 25\tan^2\theta$

$dx = 5\sec\theta\tan\theta\,d\theta.$



$\sec\theta = \frac{x}{5} = \frac{\text{HYP}}{\text{ADJ}}$

5.) Notice that $(x^2-4)^2 = [(x+2)(x-2)]^2 = (x+2)^2(x-2)^2$ so

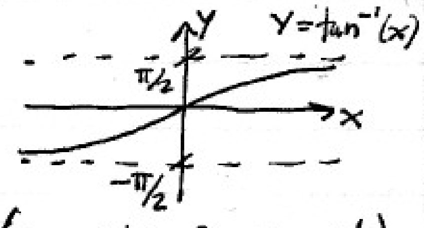
$$f(x) = \frac{x+23}{(x+2)^2(x-2)^2(x+3)(x^2+9)^2}$$

$$= \boxed{\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x-2)} + \frac{D}{(x-2)^2} + \frac{E}{x+3} + \frac{Fx+G}{x^2+9} + \frac{Hx+I}{(x^2+9)^2}}.$$

6.) a.) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$

$\displaystyle = \lim_{t\to\infty} \left(\tan^{-1}(t) - \tan^{-1}(0)\right)$

$\displaystyle = \lim_{t\to\infty} \left(\tan^{-1}(t)\right) = \boxed{\pi/2}$



(can obtain from tangent graph.)

b.) $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx$

$\displaystyle = \lim_{t\to 0^+} \left(\ln|1| - \ln|t|\right)$

$\displaystyle = \lim_{t\to 0^+} \left(-\ln(t)\right) = -(-\infty) = \boxed{\infty}$

Bonus Problems:

$$\int \csc\theta \, d\theta = -\int \frac{du}{u}$$

$u = \csc\theta + \cot\theta$

$du = (-\csc\theta\cot\theta - \csc^2\theta)d\theta$

$= -\csc\theta(\cot\theta + \csc\theta)d\theta$

$\therefore \frac{du}{u} = -\csc\theta \, d\theta.$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\csc\theta + \cot\theta| + C}$$

this one was pretty easy given we've got $\sec\theta$ on the brain.

$$\int \frac{x+5}{(x^2+4x+5)^2}dx = \int \frac{x+5}{[(x+2)^2+1]^2}dx$$

: completed square downstairs.

$$= \int \frac{u+3}{[u^2+1]^2}du$$

: $u = x+2$, $du = dx$, $x = u-2$

$$= \int \frac{u\,du}{(u^2+1)^2} + \int \frac{3du}{(u^2+1)^2}$$

$$= \int \frac{\frac{1}{2}dw}{w^2} + \int \frac{3\sec^2\theta \, d\theta}{(\sec^2\theta)^2}.$$

$w = u^2+1$	$u = \tan\theta$
$dw = 2u\,du$	$u^2+1 = \tan^2\theta + 1 = \sec^2\theta$
	$du = \sec^2\theta \, d\theta$

$$= \frac{-1}{2}\frac{1}{w} + \int \cos^2\theta \, d\theta$$

$$= -\frac{1}{2}\frac{1}{u^2+1} + \int \frac{1}{2}(1+\cos 2\theta)d\theta$$

$$= -\frac{1}{2}\frac{1}{u^2+1} + \frac{\theta}{2} + \frac{1}{4}\sin(2\theta) + C$$

$$= \frac{-1}{2}\frac{1}{x^2+4x+5} + \frac{1}{2}\tan^{-1}(u) + \frac{1}{4}\sin(2\tan^{-1}(u)) + C$$

$$= \boxed{\frac{-1}{2}\frac{1}{x^2+4x+5} + \frac{1}{2}\tan^{-1}(x+2) + \frac{1}{4}\sin(2\tan^{-1}(x+2)) + C}$$

The 20pt. Bonus

$$\int \frac{d\theta}{\sin\theta+\cos\theta+1} = \int \frac{1-\sin\theta-\cos\theta}{1-\sin\theta-\cos\theta}\,\frac{1}{1+\sin\theta+\cos\theta}\,d\theta$$

$$= \int \frac{1-\sin\theta-\cos\theta}{1-(\sin\theta+\cos\theta)^2}\,d\theta$$

$$= \int \frac{1-\sin\theta-\cos\theta}{1-2\sin\theta\cos\theta-\sin^2\theta-\cos^2\theta}\,d\theta$$

$$= \int \frac{1-\sin\theta-\cos\theta}{-2\sin\theta\cos\theta}\,d\theta$$

$$= \int \frac{-1}{2\sin\theta\cos\theta}\,d\theta + \frac{1}{2}\int\frac{d\theta}{\cos\theta} + \frac{1}{2}\int\frac{d\theta}{\sin\theta}$$

$$= \int \frac{d\theta}{\cancel{2}\sin(2\theta)} + \frac{1}{2}\int \sec\theta\, d\theta + \frac{1}{2}\int \csc\theta\, d\theta$$

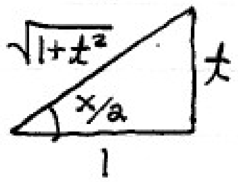
$$= \int \csc(2\theta)\,d\theta + \frac{1}{2}\int\sec\theta\, d\theta + \frac{1}{2}\int\csc\theta\, d\theta$$

$$= \boxed{-\frac{1}{2}\ln|\csc(2\theta)+\cot(2\theta)| + \frac{1}{2}\ln|\sec\theta+\tan\theta| + \frac{1}{2}\ln|\csc\theta+\cot\theta| + C}$$

the sol^n above is due to my brother, the sol^n below is due to my wife Ginny, something she learned in highschool!

this substition will allow you to integrate $\frac{1}{a\sin\theta+b\cos\theta+c}$ for any a, b, c!

$t = \tan(x/2)$ (say what?)



$$dt = \frac{1}{2}\sec^2(x/2)\,dx$$

$$dx = 2\cos^2(x/2)\,dt = \left(\frac{2}{1+t^2}\right)dt$$

$$\sin\left(\frac{x}{2}\right) = t/\sqrt{1+t^2} \quad \& \quad \cos\left(\frac{x}{2}\right) = 1/\sqrt{1+t^2}$$

$$\sin(x) = 2\sin(x/2)\cos(x/2) = \frac{2t}{\sqrt{1+t^2}}\,\frac{1}{\sqrt{1+t^2}} = 2t/(1+t^2)$$

$$\cos(x) = \cos^2(x/2)-\sin^2(x/2) = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = (1-t^2)/(1+t^2)$$

$$\int\frac{dx}{1+\sin x+\cos x} = \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\left(\frac{2}{1+t^2}\right)dt$$

$$= \int\frac{2\,dt}{1+t^2+2t+1-t^2} = \int\frac{2\,dt}{2+2t} = \ln|t+1|+C$$

$$= \boxed{\ln|\tan(x/2)+1| + C}$$