YOU MAY KEEP THE TEST PAPER, PUT NAME ON WORK.

Instructor: James Cook , MA 241-003 Calculus II, Test II: Applications of Integration, July 16, 2007

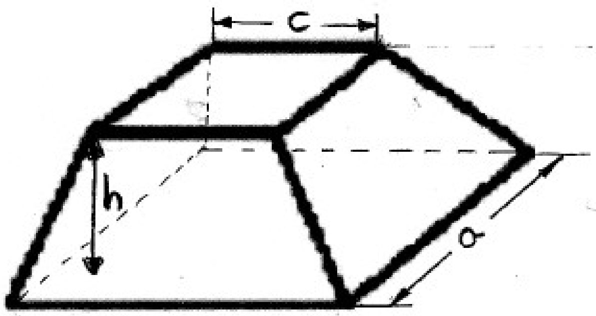
Directions: Show your work, if you doubt that you've shown enough detail then ask. No electronic aids of any sort are permitted modulo your watch and a scientific calculator.

1. (15pts) Show that a circle with radius R has area $A = \pi R^2$. Your solution should include a careful graph and a sketch of the typical infinitesimal rectangle.

2. (15pts) Find the area of the region bounded by $x + y = 0$ and $x + y^2 = 2$. Your solution should include a careful graph and a sketch of the typical infinitesimal rectangle. Use algebra to find any intersection points of interest.

3. (15pts) Find volume of solid obtained by rotating the region bounded by $y = x$ and $y = \sqrt{x}$ around $x = 2$. Your solution should include a graph which indicates the typical infinitesimal washer. Again algebra should be used to find points of intersection.

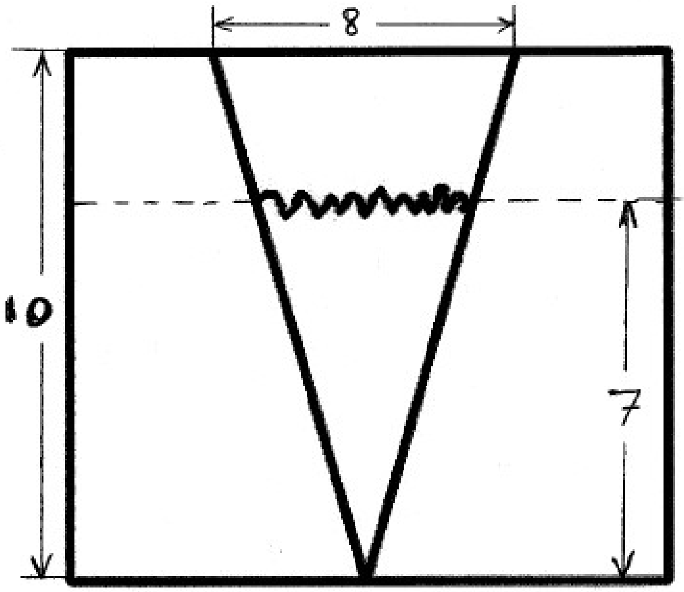
4. (15pts) Find volume of a square pyramid which has its top cut off at height h. Suppose that the each side of the base of pyramid has length a. The top of the shape is coplanar to the base and the top forms a square with sides of length c. Your answer will be in terms of the constants a, c and h.



5. (10pts) Show that the circumference of a circle of radius R is $s = 2\pi R$. Use the arclength formula applied to the parametric equations for the circle; $x = R\cos(t)$ and $y = R\sin(t)$ for $0 \leq t \leq 2\pi$.

6. (15pts) Find the magnitude of work required to lift a rope of length $25m$ with a rock of mass $100kg$ tied to the end. Suppose that the rope has total mass of 50kg which is uniformly distributed. To begin you should define any needed variable and then calculate the infinitesimal work needed to lift the system a tiny distance. You may leave the answer in terms of the acceleration due to gravity; g.

7. (15pts) Find the hydrostatic force against the triangular gate pictured below. You may leave your answer in terms of the density of water ρ and g. Be sure to include a diagram of the typical infinitesimal strip with area dA on which the water places an infinitesimal force dF.

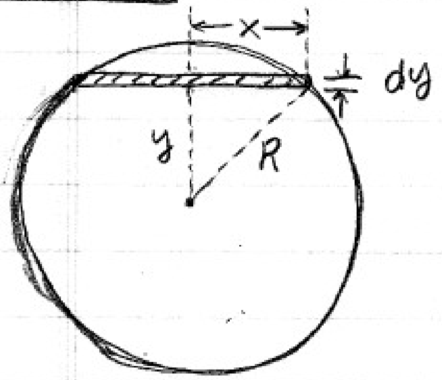


8. BONUS:(15pts) Calculate the following,
(5pts): the volume of the solid obtained from rotating region bounded by $y = \sin(x)$ and $y = 0$ and $x = 0$ and $x = \pi$ around the y-axis. Use the method of cylindrical shells.

(10pts): $\int \frac{1}{\sin(x) + \cos(x)}\, dx$,
WARNING: don't even try this last one unless you have everything else completely finished and checked, this one is nontrivial.

TEST II SOLUTION, ma 241-003, summer II 2007

PROBLEM ONE



$$dA = 2x\,dy \quad \text{, note } x^2 + y^2 = R^2$$

$$dA = 2\sqrt{R^2 - y^2}\,dy$$

$$A = \int_{-R}^{R} 2\sqrt{R^2-y^2}\,dy \longrightarrow \boxed{\begin{array}{l} y = R\sin\theta \\ dy = R\cos\theta\,d\theta \\ R^2 - y^2 = R^2\cos^2\theta \\ y = R \Rightarrow \theta = \frac{\pi}{2} \\ y = -R \Rightarrow \theta = \frac{-\pi}{2} \end{array}}$$

$$= \int_{-\pi/2}^{\pi/2} 2R^2\cos^2\theta\,d\theta$$

$$= R^2 \int_{-\pi/2}^{\pi/2} (1 + \cos(2\theta))\,d\theta$$

$$= R^2 \left[\theta + \tfrac{1}{2}\cancel{\sin(2\theta)}_{0} \right|_{-\pi/2}^{\pi/2} =$$

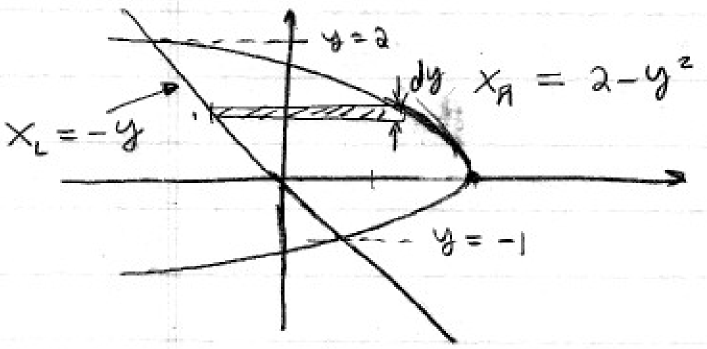
$$= R^2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \boxed{\pi R^2}.$$

PROBLEM TWO

$x + y = 0$ and $x + y^2 = 2$ bound a region, find its area. Note $x = x \Rightarrow -y = 2 - y^2$

$$\Rightarrow y^2 - y - 2 = 0 = (y-2)(y+1)$$

$\therefore$ $y = 2$ and $y = -1$ at intersection.



$$dA = (x_R - x_L)\,dy$$

$$dA = (2 - y^2 + y)\,dy$$

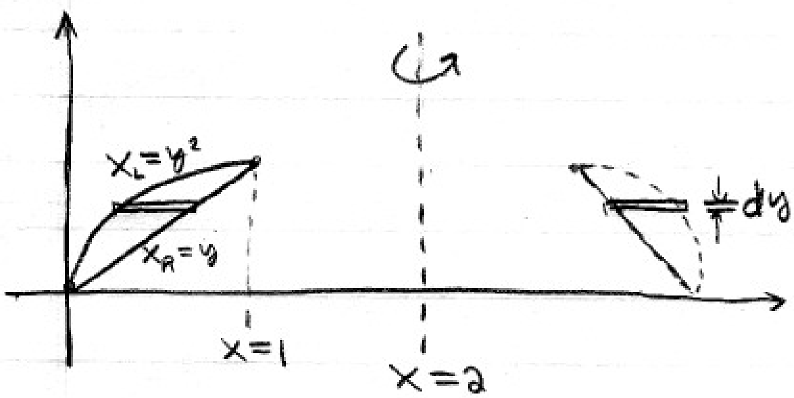
$$A = \int_{-1}^{2} (2 - y^2 + y)\,dy = \left[2y - y^3/3 + y^2/2\right]_{-1}^{2}$$

$$= [4 - 8/3 + 2] - [-2 + 1/3 + 1/2]$$

$$= 8 - 8/3 - 1/3 - 1/2 = 9/2.$$

$$= \boxed{4.5}$$

PROBLEM THREE Notice $y=x$ and $y=\sqrt{x}$ bounds a region. Find volume of solid obtained by rotating region around $x=2$.



$r_{in} = 2 - x_R$
$r_{out} = 2 - x_L$

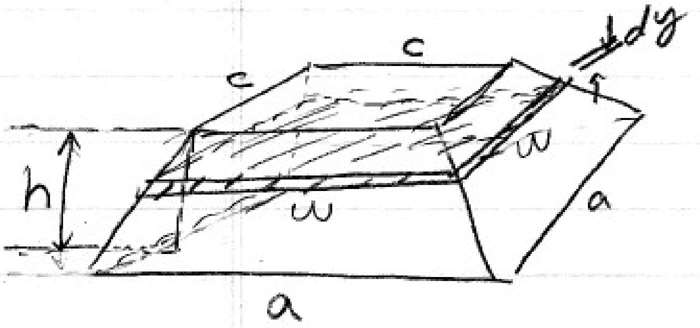
$$
\begin{aligned}
dV &= \pi\left(r_{out}^2 - r_{in}^2\right)dy \\
&= \pi\left((2-y^2)^2 - (2-y)^2\right)dy \\
&= \pi\left(\cancel{4} - 4y^2 + y^4 - \cancel{4} + 4y - y^2\right)dy \\
&= \pi\left(y^4 - 5y^2 + 4y\right)dy
\end{aligned}
$$

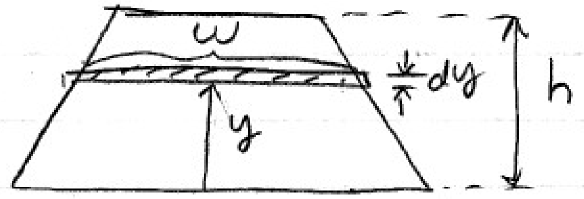
Points of intersection, $y = y \Rightarrow x = \sqrt{x}$

$$
\begin{aligned}
&\Rightarrow x^2 = x \Rightarrow x^2 - x = x(x-1) = 0 \\
&\Rightarrow x=0 \text{ and } x=1 \\
&\Rightarrow y=0 \text{ and } y=1
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_0^1 \pi\left(y^4 - 5y^2 + 4y\right)dy \\
&= \pi\left(\frac{1}{5} - \frac{5}{3} + \frac{4}{2}\right) \\
&= \pi\left(\frac{3 - 25 + 30}{15}\right) \\
&= \boxed{8\pi/15}.
\end{aligned}
$$

PROBLEM FOUR





$$dV = w^2 dy$$
$$dV = \left[\left(\frac{c-a}{h}\right)y + a\right]^2 dy$$

$w = my + b$

$a = m(0) + b \therefore \; b = a$

$c = mh + a \therefore \; m = \frac{c-a}{h}$

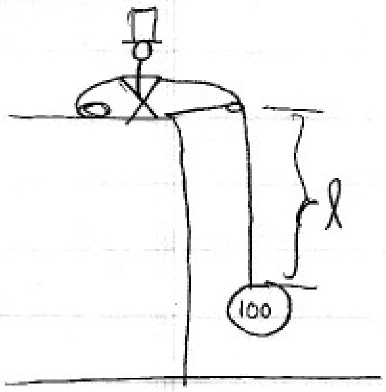
$$V = \int_0^h \left[\left(\frac{c-a}{h}\right)y + a\right]^2 dy$$
$$= \int_0^h \left[\left(\frac{c-a}{h}\right)^2 y^2 + 2a\left(\frac{c-a}{h}\right) + a^2\right] dy$$
$$= \frac{1}{3}\left(\frac{c-a}{h}\right)^2 h^3 + a\left(\frac{c-a}{h}\right)h^2 + a^2 h$$
$$= \boxed{\frac{1}{3}(c-a)^2 h + a(c-a)h + a^2 h}$$

PROBLEM FIVE

$x = R\cos t$
$y = R\sin t$

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^{2\pi} \sqrt{(-R\cos t)^2 + (R\sin t)^2}\, dt$$
$$= \int_0^{2\pi} \sqrt{R^2(\cos^2 t + \sin^2 t)}\, dt$$
$$= \int_0^{2\pi} R\, dt = Rt\Big|_0^{2\pi} = \boxed{2\pi R}.$$

PROBLEM SIX

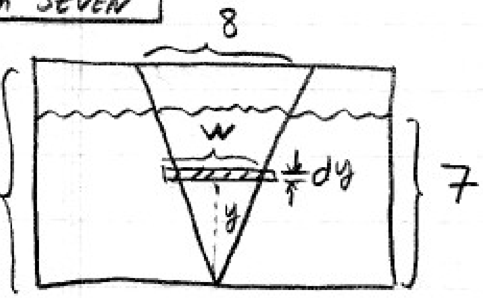


$$F_{gravity} = m_{rock}\, g + m_{rope}\, g$$
$$= 100g + \left(\frac{50}{25} l\right) g$$
$$= g(100 + 2l).$$

$$dW = F_{gravity}\, dl = g(100 + 2l)\, dl$$

$$W = \int_0^{25} g(100 + 2l)\, dl = g(100l + l^2 \Big|_0^{25}$$
$$= \boxed{g((100)(25) + (25)^2)}$$

PROBLEM SEVEN

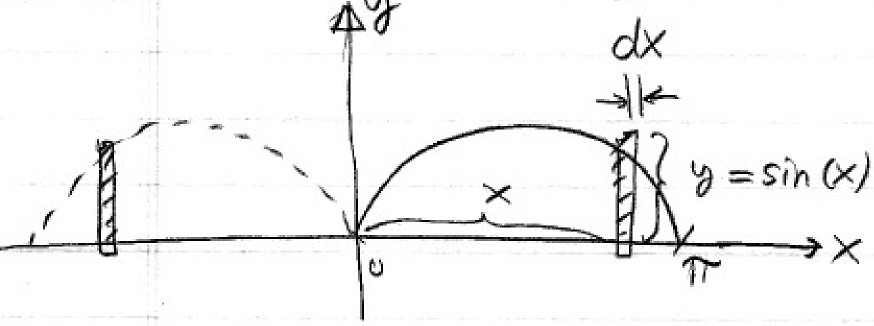


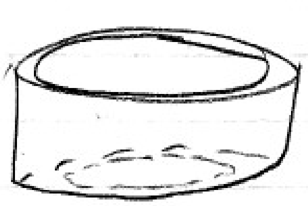
$$dA = w\, dy$$
$$w = my + b$$
$$y = 0 \Rightarrow w = b = 0$$
$$y = 10 \Rightarrow 10m = 8 \therefore m = \frac{4}{5}$$
$$\therefore \; dA = \frac{4}{5} y\, dy$$

$$dF = P\, dA = (\rho g d) dA = \rho g (7 - y) \frac{4}{5} y\, dy$$

$$F = \int_0^7 \frac{4\rho g}{5} (7y - y^2)\, dy$$
$$= \frac{4\rho g}{5} \left(\frac{7y^2}{2} - \frac{y^3}{3} \Big|_0^7 \right.$$
$$= \frac{4\rho g}{5} \left(\frac{7^3}{2} - \frac{7^3}{3} \right) = \boxed{\frac{2\rho g}{15} (7^3)}.$$

<u>Bonus</u>: Use method of cylindrical shells,





$$dV = 2\pi x y \, dx = 2\pi x \sin(x) \, dx$$

$$V = \int_0^{\pi} 2\pi x \sin(x) \, dx$$

$u = 2\pi x$	$dv = \sin(x)dx$
$du = 2\pi dx$	$v = -\cos(x)$

$$= -2\pi x \cos(x) \Big|_0^{\pi} + 2\pi \int_0^{\pi} \cos(x) dx$$

$$= 2\pi^2 + 2\pi \left(\sin(x) \Big|_0^{\pi} \right) = \boxed{2\pi^2}$$

$\int \frac{dx}{\sin(x) + \cos(x)}$ can be calculated by the substitution $t = \tan(x/2)$.

the process is similar to $\int \frac{1}{1 + \sin(x) + \cos(x)} dx$ that was a previous bonus problem, the integral in t that is obtained is not as pretty but is still just an elementary (although messy) partial fractions problem.

- there is also an easier solⁿ w/o the $t = \tan(x/2)$ trick.