Ma 241.07 Test II

Febuary 28, 2005

1 Improper integration(20pts)

Find if the integrals below converge or diverge. If it converges find the value to which it converges. Show your work.

(a) $\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty} \int_0^t \underset{u}{x}\,\underset{dv}{e^{-x}dx}$

$= \lim_{t\to\infty} \left(-xe^{-x}\Big|_0^t + \int_0^t e^{-x}dx\right)$

$= \lim_{t\to\infty} \left(-te^{-t} - e^{-t} + e^0\right)$

$= \lim_{t\to\infty} (-t/e^t) - \underbrace{\lim_{t\to\infty}(e^{-t})}_{0} + 1$

$\overset{L}{=} \underbrace{\lim_{t\to\infty}\left(\frac{-1}{e^t}\right)}_{0} + 1$

$(\frac{\infty}{\infty})$

$= \boxed{1}$

this was #5 of §5.10 webassign

(b) $\int_1^9 \frac{1}{\cancel{(x-9)}^{\frac{3}{2}}}\,dx$ $9-x$

Let $\boxed{\begin{matrix} u = 9-x \\ du = -dx \end{matrix}} \longrightarrow \int \frac{1}{(9-x)^{3/2}}dx = \int u^{-3/2}(-du) = -\frac{u^{-1/2}}{-1/2} + C = \frac{2}{\sqrt{9-x}} + C$

$\int_1^9 \frac{dx}{(9-x)^{3/2}} = \lim_{t\to 9^-} \int_1^t \frac{dx}{(9-x)^{3/2}}$

$= \lim_{t\to 9^-} \left(\frac{2}{\sqrt{9-x}}\Big|_1^t\right)$

$= \lim_{t\to 9^-} \left(\underbrace{\frac{2}{\sqrt{9-t}}}_{\infty} - \frac{2}{\sqrt{8}}\right)$ $\therefore$ the integral diverges.

this is similar to E12 in notes

2 Geometry: arclength, area and volume (35pts)

Recall that a circle of radius a centered at the origin is $x^2 + y^2 = a^2$.

(a)(10pts) Show that the circumference of a circle of radius a is $2\pi a$.

$X = r\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$Y = r\sin\theta$ $\quad$ are the parametric eq^{n}'s of circle

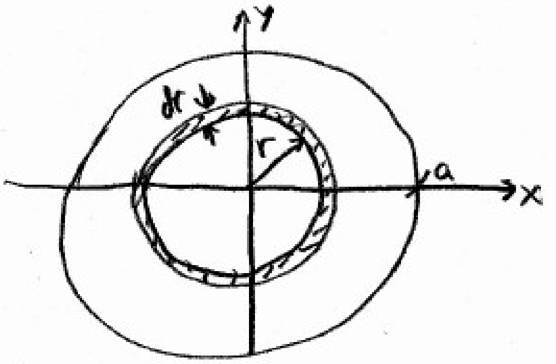
$\frac{dx}{d\theta} = -r\sin\theta$ and $\frac{dy}{d\theta} = r\cos\theta$ thus

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} r\sqrt{\underbrace{\sin^2\theta + \cos^2\theta}_{1}}\, d\theta = \int_0^{2\pi} r\, d\theta = \boxed{2\pi r}$$

(b)(10pts) Show that the area of a circle of radius a is πa^2.



the area of each washer is

$$dA = 2\pi r\, dr$$

Thus

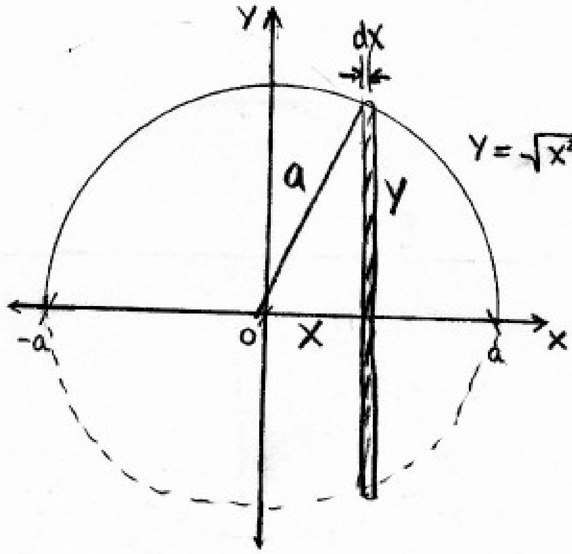
$$A = \int_0^a 2\pi r\, dr$$

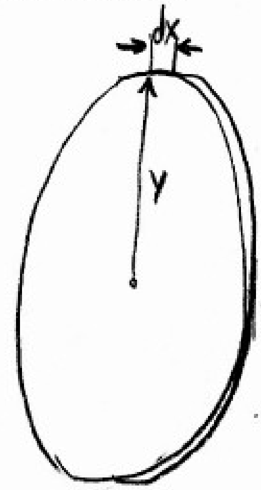
$$= 2\pi \frac{r^2}{2}\Big|_0^a$$

$$= \boxed{\pi a^2}$$

- I covered this method during the review day. If you used the other method compare to the online notes where it is worked out. I was hoping everybody would use this easier sol^{n} for the test.

(c)(15pts) Define a region R in the $(xy)-plane$ to be the set of points bounded by the curve $y = \sqrt{x^2 - a^2}$ and $y = 0$. Find the volume of the solid obtained from rotating the region R around the x axis. What is the name of this solid ?





$$A = \pi y^2$$

$$dV = \pi y^2 dx$$
$$= \pi (a^2 - x^2) dx$$

Then we sum the volumes of the disks from $x = -a$ to $x = a$ by integrating

$$V = \int_{-a}^{a} \pi (a^2 - x^2) dx$$

$$= \pi \left(a^2 x - \frac{1}{3} x^3 \right|_{-a}^{a}$$

$$= \pi \left[\left(a^2(a) - \frac{1}{3} a^3\right) - \left(a^2(-a) - \frac{1}{3}(-a)^3\right) \right]$$

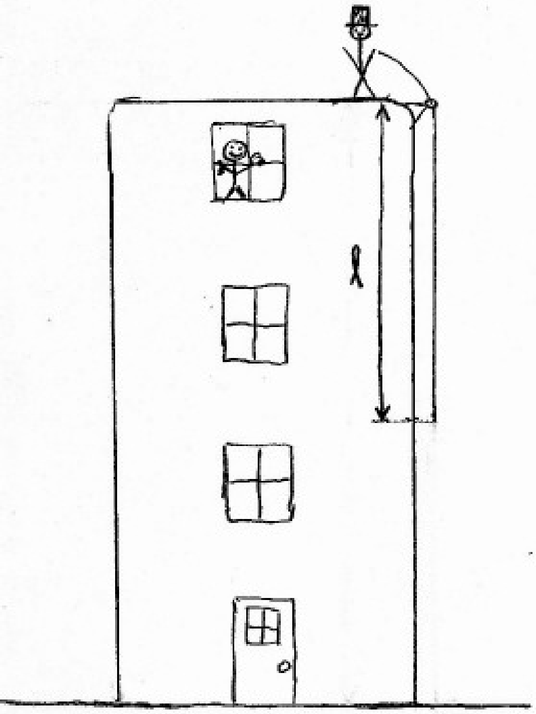
$$= \pi \left[\frac{2}{3} a^3 + \frac{2}{3} a^3 \right]$$

$$= \boxed{\frac{4}{3} \pi a^3}$$

THIS was an example in the notes.
& It was explicitly listed on review sheet

3 Physics (40pts)

(a)(20pts) Consider a cable with mass M and length l. Assume that the cable is intially hanging from a rooftop. Also assume the cable's mass is uniformly distributed along it's length. Calculate the magnitude of work required to lift the whole cable to the rooftop. Your answer should involve M , l and the acceleration due to gravity g.



Let Y = length of cable hanging.
Clearly $0 \le Y \le l$. That is $Y = l$ at the beginning & $Y = 0$ after we pull up the cable.

$$F = mg = \frac{M}{l} y g$$

The mass (m) being lifted is a fnct. of Y.

$$W = \int_l^0 mg\, dY$$

$$= \int_l^0 \frac{M}{l} y g\, dY$$

$$= \frac{Mg}{l} \int_l^0 y\, dY$$

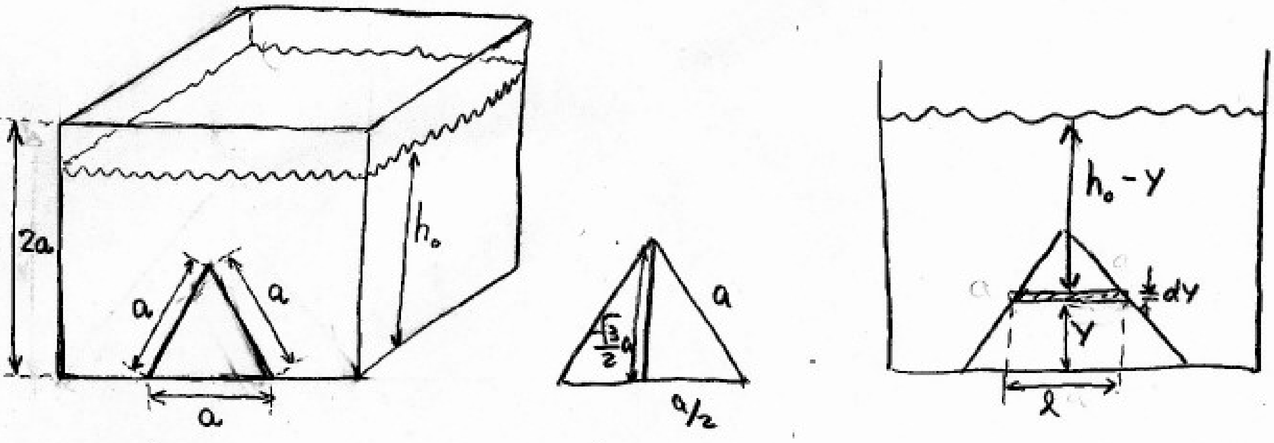
$$= \frac{Mg}{l} \left(\frac{1}{2} y^2 \Big|_l^0 \right)$$

$$= -\frac{Mgl^2}{2l} \Rightarrow \boxed{|W| = \frac{1}{2} Mgl}$$

- Incidentally in a physics course we could find this by viewing the cable as a single particle of mass M located at the center of mass. Of course you weren't supposed to do it that way. The method above was an example in notes.

• This is a webassign problem with an easier region to integrate. I had hoped you would know how to relate dA to y from the pyramid, cone and other such problems.

(b)(20pts) Consider the tank of water pictured below. The tank is filled to a height h_0. What is the force due to water pressure on the triangular window pictured below ? You may assume the ρ is the density of water. Your answer should involve ρ , h_0 , a and the acceleration due to gravity g.



What is dA for the strip at height y?

$$dA = \ell\, dy$$

Notice the length of the strip varies linearly with y so,

$$\ell = my + b$$

$$y = 0 \Rightarrow \ell = \boxed{a = b}$$

$$y = \frac{\sqrt{3}}{2}a \Rightarrow \ell = 0 = m\frac{\sqrt{3}}{2}a + b = \frac{\sqrt{3}a}{2}m + a \quad \therefore \quad \boxed{m = \frac{-2}{\sqrt{3}}}$$

Thus $\boxed{\ell = \frac{-2}{\sqrt{3}}y + a}$. Now the $P = \frac{dF}{dA}$ so $dF = P\,dA$

where $P = \rho g(h_0 - y)$ and $dA = \left(-\frac{2}{\sqrt{3}}y + a\right)dy$ hence,

$$F = \int_0^{\frac{\sqrt{3}a}{2}} \rho g (h_0 - y)\left(-\frac{2}{\sqrt{3}}y + a\right)dy$$

$$= \rho g \int_0^{\frac{\sqrt{3}}{2}a} \left(h_0 a - \left(a + \frac{2}{\sqrt{3}}h_0\right)y + \frac{2}{\sqrt{3}}y^2\right)dy$$

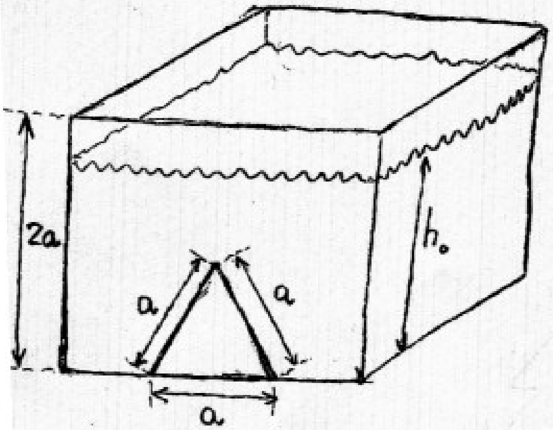
$$= \rho g \left[h_0 a y - \frac{1}{2}\left(a + \frac{2}{\sqrt{3}}h_0\right)y^2 + \frac{2}{3\sqrt{3}}y^3 \right|_0^{\frac{\sqrt{3}}{2}a}$$

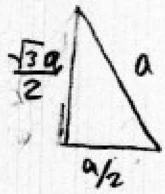
$$\frac{2}{3\sqrt{3}}\left(\frac{\sqrt{3}a}{2}\right)^3 = a^3/4$$

$$= \boxed{\rho g \left[\frac{\sqrt{3}}{2}h_0 a^2 - \frac{1}{2}\left(a + \frac{2}{\sqrt{3}}h_0\right)\frac{3a^2}{4} + \frac{1}{4}a^3 \right]}$$

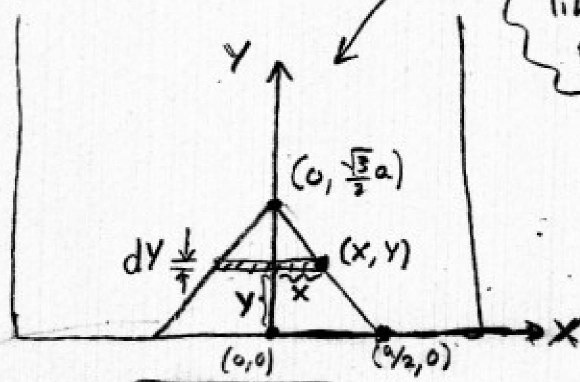
Alternatively we could use the line fitting idea to relate dA to the height of strip Y.

(b)(20pts) Consider the tank of water pictured below. The tank is filled to a height h_0. What is the force due to water pressure on the triangular window pictured below ? You may assume the ρ is the density of water. Your answer should involve ρ , h_0 , a and the acceleration due to gravity g.





choosing coordinates like other problem.



$$\boxed{dA = 2x\,dY} \quad \text{for} \quad 0 \le Y \le \frac{\sqrt{3}a}{2}$$

We can relate X & Y by finding the eq^{n} of line that connects the points $(0,\frac{\sqrt{3}}{2}a)$ and $(a,0)$.

$$X=0\ ,\ Y=\tfrac{\sqrt{3}}{2}a \Rightarrow \tfrac{\sqrt{3}}{2}a = m(0)+b \ \therefore \ \boxed{b=\tfrac{\sqrt{3}}{2}a}$$ (clear from picture as well.

$$X=a/2\ ,\ Y=0 \Rightarrow 0 = m(\tfrac{a}{2}) + \tfrac{\sqrt{3}}{2}a \ \therefore \ \boxed{m=-\sqrt{3}}$$ (also clear from picture.

Thus $Y = -\sqrt{3}X + \frac{\sqrt{3}}{2}a$ now we must solve for X

$$\boxed{X = -\tfrac{1}{\sqrt{3}}Y + a/2} \Rightarrow \boxed{dA = \left(\tfrac{-2}{\sqrt{3}}Y + a\right)dY}$$

Same as last time, we just found it by slightly different method.

As in other sol^{n} $P = \rho g d = \rho g(h_0 - Y)$ then since $dF = P\,dA$

$$F = \int_0^{\frac{\sqrt{3}a}{2}} \rho g(h_0 - Y)\left(\tfrac{-2}{\sqrt{3}}Y + a\right)dY$$

$$= \rho g \int_0^{\frac{\sqrt{3}a}{2}} \left(h_0 a - (a + \tfrac{2}{\sqrt{3}}h_0)Y + \tfrac{2}{\sqrt{3}}Y^2\right)dY$$

$$= \rho g \left[h_0 a Y - \tfrac{1}{2}(a + \tfrac{2}{\sqrt{3}}h_0)Y^2 + \tfrac{2}{3\sqrt{3}}Y^3\right|_0^{\frac{\sqrt{3}}{2}a}$$

$$= \boxed{\rho g \left[\tfrac{\sqrt{3}}{2}h_0 a^2 - \tfrac{1}{2}(a + \tfrac{2}{\sqrt{3}}h_0)\tfrac{3a^2}{4} + \tfrac{1}{4}a^3\right]}$$

didn't expect you to simplify just include so you can compare answers.

$$= \rho g\left[\left(\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{4}\right)h_0 a^2 + \left(\tfrac{1}{4} - \tfrac{3}{8}\right)a^3\right] = \boxed{\rho g\left(\tfrac{\sqrt{3}}{4}h_0 a^2 - \tfrac{1}{8}a^3\right)}$$

4 PROBABILITY(10PTS)

(a)(10pts) Show that the $f(x)$ defined below is a probability distribution

$$f(x) = \begin{cases} \frac{4}{\pi}\frac{1}{1+x^2}, & \text{if } x \geq 1; \\ 0, & \text{if } x < 1; \end{cases}$$

Notice first that $f(x)$ is non-negative since $1+x^2 \geq 1$ So there is no way for $f(x) = \frac{4}{\pi}\frac{1}{1+x^2}$ to be negative.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{1} 0 \cdot dx}_{0} + \int_{1}^{\infty} \frac{4}{\pi}\frac{1}{1+x^2}\,dx \quad \text{: by the def}^{\text{n}} \text{ of } f(x)$$

$$= \lim_{t\to\infty} \int_{1}^{t} \frac{4}{\pi}\frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \left(\frac{4}{\pi} \tan^{-1}(x) \Big|_{1}^{t} \right)$$

$$= \lim_{t\to\infty} \left(\frac{4}{\pi}\tan^{-1}(t) - \frac{4}{\pi}\tan^{-1}(1) \right)$$

$$= \frac{4}{\pi}\Big(\underbrace{\lim_{t\to\infty} \tan^{-1}(t)}_{\frac{\pi}{2}}\Big) - \frac{4}{\pi}\frac{\pi}{4}\Big)$$

$$= \frac{4}{2} - \frac{4}{4}$$

$$= \boxed{1}$$

Thus $f(x)$ is a probability distribution since

① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$.