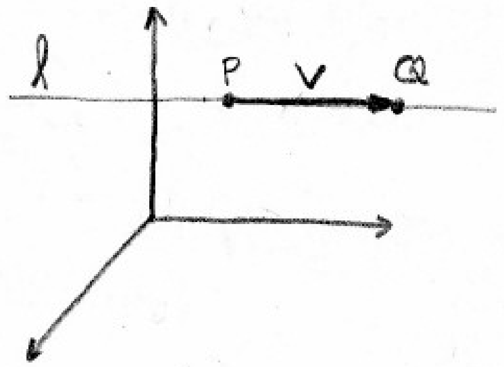
LINES & PLANES IN 3-d

To begin we introduce the notion of a parametrized line. The parameter "t" or "s" (or whatever you prefer) is an extra variable introduced to give a convenient description of the line. The parameter could be chosen to be arclength or perhaps x, y or z. There is much freedom in the choice.

<u>The standard trick</u>: well its not much of a trick really, but to parametrize a line ℓ that passes through points P and Q we construct



$$V = Q - P$$

then write

$$r(t) = P + tV$$

this has $r(0) = P$ and $r(1) = Q$.

notice the parametrization gives the line ℓ a direction, we can say ℓ is oriented if we insist on a direction for its parametrization. Otherwise, there are two choices $(\longrightarrow)$ or $(\longleftarrow)$.

Def^{n}/ Suppose that $r_o = \langle x_o, y_o, z_o \rangle$ and $V = \langle a, b, c \rangle$ then $r(t) = r_o + tV$ is a line with initial point r_o and <u>direction</u> V. The parametric eq^{n}'s for $r(t) = \langle x(t), y(t), z(t) \rangle$ are simply found by looking at $r(t) = r_o + tV$,

$$x(t) = x_o + at$$
$$y(t) = y_o + bt$$
$$z(t) = z_o + ct$$

(note a, b, c are "direction #'s" for the line)

E10 find parametric eq^{n}'s of line with direction $\langle 1, 0, 1 \rangle = V$ and initial point $r_o = \langle \pi, \pi, \pi \rangle$. Well $r(t) = r_o + tV$ so $r(t) = \langle \pi, \pi, \pi \rangle + t\langle 1, 0, 1 \rangle = \langle \pi + t, \pi, \pi + t \rangle$

$$\boxed{\begin{aligned} x &= \pi + t \\ y &= \pi \\ z &= \pi + t \end{aligned}}$$

Def^{n} If L is a line in the $v = \langle a, b, c\rangle$ direction that passes through $r_o = \langle x_o, y_o, z_o\rangle$ then the <u>symmetric</u> <u>eq^{n}'s</u> for L are $(a, b, c \neq 0)$

$$\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c}$$

pragmatically I don't use this formula, rather I usually just solve the parametric eq^{n}'s for t.

E11 Suppose $L : r(t) = \langle 3-t,\ t+5,\ 2t+8\rangle$ then

$$\left.\begin{aligned} x &= 3-t &&\Rightarrow t = 3-x \\ y &= t+5 &&\Rightarrow t = 5-y \\ z &= 2t+8 &&\Rightarrow t = \tfrac{1}{2}(z-8) \end{aligned}\right\} \Rightarrow t = \boxed{3-x = 5-y = \tfrac{1}{2}(z-8)}$$

Ok, to be more picky we should write $\frac{x-3}{-1} = \frac{y-5}{-1} = \frac{z-8}{2}$ then we can identify $v = \langle -1, -1, 2\rangle$ and $r_o = \langle 3, 5, 8\rangle$ of course those facts are obvious from $r(t)$ to begin.

E12 In **E10** we had $v = \langle 1, 0, 1\rangle$ so $b = 0$, clearly we cannot divide by b! So the sort-of symmetric eq^{n}'s are

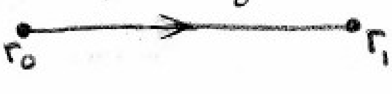
$$t = \boxed{x - \pi = z - \pi \quad \text{and} \quad y = \pi}$$

<u>Remark:</u> the nice thing about parametric eq^{n}'s for a line is that we <u>always</u> can write $r(t) = r_o + tv$, as opposed to the phenomenon we encounter in **E12** showing the symmetric eq^{n}'s can only be written for a certain subclass of all lines. I will by default use parametric description for our lines. Also later when we consider motion the t will be identified as time and $r(t)$ has a nice physical interpretation.

<u>LINE SEGMENTS:</u> are easy. You just restrict the domain of t so that it cuts-off the rest of the line. The line segment from $r_o = (x_o, y_o, z_o)$ to $r_1 = (x_1, y_1, z_1)$ is

$$\boxed{r(t) = (x_o, y_o, z_o) + t(x_1 - x_o,\ y_1 - y_o,\ z_1 - z_o) \qquad 0 \leq t \leq 1}$$

Check it out, $r(0) = r_o$ and $r(1) = r_1$. Again the parametrization gives it an orientation



arrow along increasing t

Parallel & Skew Lines

Let L_1 go in the V_1-direction and L_2 in the V_2-direction. Then lines L_1 & L_2 are parallel iff $V_1 = kV_2$ for $k \neq 0$. The lines L_1 & L_2 are skew iff they do not intersect.

$\boxed{E13}$ Show $L_1 : r_1(t) = \langle 3+t, 2+t, 1+t \rangle$ and $r_2(t) = \langle 3t, 3t, 3t+6 \rangle$ are parallel. Well notice

$$r_1(t) = \langle 3, 2, 1 \rangle + t\langle 1, 1, 1 \rangle = r_1(0) + tV_1$$

$$r_2(t) = \langle 0, 0, 6 \rangle + t\langle 3, 3, 3 \rangle = r_2(0) + tV_2$$

Clearly $V_2 = 3V_1$ thus L_1 & L_2 are parallell.

$\boxed{E14}$ Show $L_1 : r_1(t) = \langle 1+t, 3t-2, 4-t \rangle$ and $L_2 : r_2(t) = \langle 2t, 3+t, 4t-3 \rangle$ are skew. To be fair we should not check if $\exists t$ such that $r_1(t) = r_2(t)$, because they might intersect at different t. So introduce s for r_2. Examine $r_1(t) = r_2(s)$. This gives

$$\begin{aligned} 1+t &= 2s \\ 3t-2 &= 3+s \\ 4-t &= 4s-3 \end{aligned}$$

$$3t = 6s-3 = 3+s+5 \Rightarrow 5s = 11 \therefore \underline{s = 11/5}$$

$$3t = 12-12s+9 = 3+s+5 \Rightarrow 13s = 13 \therefore \underline{s = 1}$$

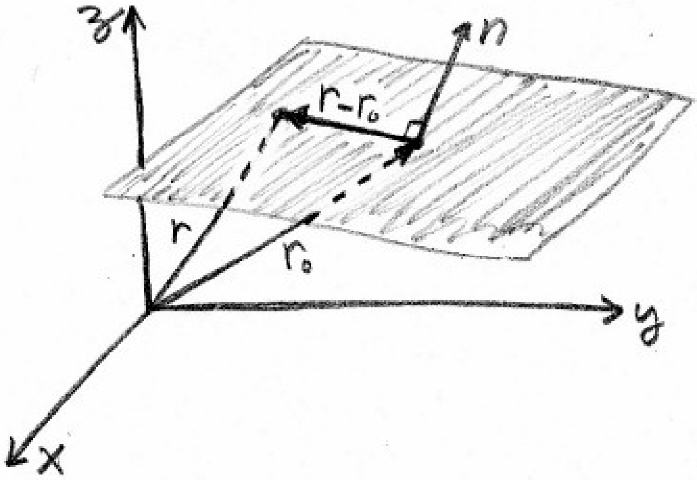
Therefore since $1 \neq 11/5$ it is seen these eq^{n}'s have no sol^{n} hence $\nexists\, s,t$ such that $r_1(t) = r_2(s)$ $\therefore$ L_1 & L_2 are skew.

Planes in $\mathbb{R}^3$

A plane is specified by a point $r_0 = \langle x_0, y_0, z_0 \rangle$ and a vector $n = \langle a, b, c \rangle$ which is orthogonal to the plane. The plane is defined to be the set of all points $r = (x, y, z) \in \mathbb{R}^3$ such that

$$\boxed{n \cdot (r - r_0) = 0}$$ <u>vector eq$^{\underline{n}}$ of plane</u>

the vector $r - r_0$ lies in the plane, I'll try to draw it



the plane is not finite, I just draw it that way. It goes on and on.

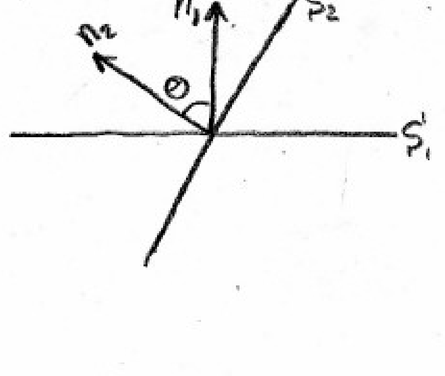
Now if $n = \langle a, b, c \rangle$ then the plane is given by

$$n \cdot (r - r_0) = \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$
$$= \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

- the scalar eq$^{\underline{n}}$'s for the plane through (x_0, y_0, z_0) with normal $\langle a, b, c \rangle$.

We say two planes are parallel if their normal vectors n_1 and n_2 are parallel, otherise the angle between the normal vectors defines the angle between the planes. (Your book says acute angle but that seems impossible w/o redefining certain normals...)

(sideview)

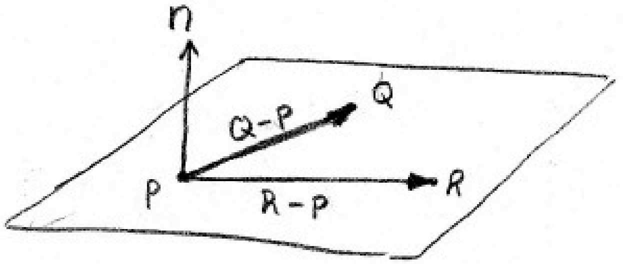


S_1 a plane with normal n_1

S_2 a plane with normal n_2

$$\Theta = \underset{\text{planes}}{\text{angle between}} = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1||n_2|}\right)$$

$\boxed{E15}$ Find eq^{n}'s of plane possessing points $(0,0,0)$, $(0,1,0)$ and $(0,1,1)$. Let me draw a picture, define



$$P = (0,0,0)$$
$$Q = (0,1,0) = \hat{j}$$
$$R = (1,1,1) = \hat{i} + \hat{j} + \hat{k}$$

$$Q - P = Q.$$
$$R - P = R.$$

$$n = (Q-P)\times(R-P) = \hat{j}\times(\hat{i}+\hat{j}+\hat{k}) = \hat{j}\times\hat{i} + \hat{j}\times\hat{k} = \langle 1, 0, -1\rangle.$$

Thus the eq^{n}'s of the plane are (using $P = r_o$) $\boxed{x - z = 0}$

<u>Remark:</u> remember the hwk sol^{n} has nastier problems.

$\boxed{E16}$ Find plane through $(1,2,3)$ with normal parallel to the intersection line of the planes

$$x + y + z = 10 \quad \text{and} \quad 2x + 3y + z = 20.$$

<u>General Principle</u>: intersection is where both eq^{n}'s are true, to quantify it pick something in common and equate, here z is a natural choice,

$$z = 10 - x - y = 20 - 2x - 3y$$

$$\Rightarrow 2y + x = 20 - 10 \Rightarrow x = 10 - 2y.$$

So we can parametrize the line by the y-coordinate,

$$\begin{aligned} r(y) &= \langle 10 - 2y,\ y,\ 10 - x - y\rangle, \quad x = 10 - 2y \\ &= \langle 10 - 2y,\ y,\ y\rangle \\ &= \langle 10, 0, 0\rangle + y\langle -2, 1, 1\rangle. \end{aligned}$$

The direction of the line of intersection is $\langle -2, 1, 1\rangle$ our plane is thus

$$\boxed{-2(x-1) + (y-2) + (z-3) = 0}$$

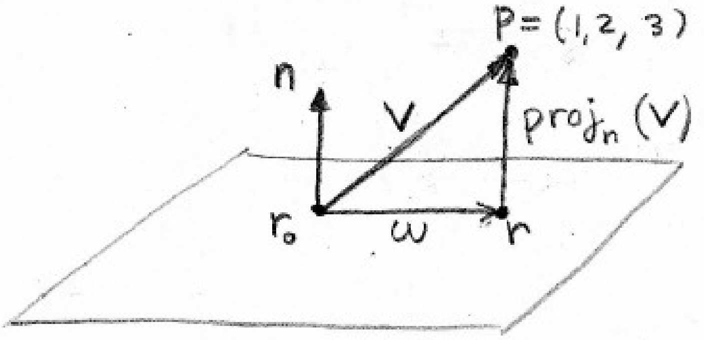
<u>Remark:</u> there is a shorter more efficient sol^{n}. Just pick to points on line, don't try to find $r(y)$.

Remark: the are many convoluted ways to give you information to find the eq$^{\underline{ns}}$ of a plane. What I always do is look for a point and some way to get the normal. Your homework explores various set-ups. You should try to understand the concept, remembering a dozen different formulas is not the best way.

$\boxed{E17}$ I return to the question at the bottom of 239, If we have the plane $x - y + 10z = 10$ then what point on plane is closest to $P = (1,2,3)$? Lets pick a point on the plane. Choose,

$$x = 0,\ y = 0 \Rightarrow 10z = 10 \therefore z = 1 \Rightarrow r_o = (0,0,1)$$

the normal to the plane is $n = \langle 1, -1, 10\rangle$ so we can draw a schematic picture of what we have and want



$V = P - r_o = \langle 1, 2, 2\rangle$

$r =$ the desired point

$r = r_o + w = P - \text{proj}_n(V)$

notice $|n| = \sqrt{102}$

$$\begin{aligned} r &= P - \text{proj}_n(V) \\ &= (1,2,3) - \frac{n \cdot V}{|n|^2}\, n \\ &= \langle 1,2,3\rangle - \tfrac{1}{102}\big(\langle 1,-1,10\rangle \cdot \langle 1,2,2\rangle\big)\langle 1,-1,10\rangle \\ &= \langle 1,2,3\rangle - \left(\tfrac{19}{102}\right)\langle 1,-1,10\rangle \\ &= \tfrac{1}{102}\langle 102-19,\ 204+19,\ 306-190\rangle \\ &= \tfrac{1}{102}\langle 83, 223, 116\rangle \ \therefore\ \boxed{\left(\tfrac{83}{102}, \tfrac{223}{102}, \tfrac{116}{102}\right) \text{ closest point.}} \end{aligned}$$

As a check is r on the plane? $\frac{83}{102} - \frac{223}{102} + \frac{1160}{102} = \frac{1020}{102} = 10$, yep.

PROJECTION ONTO PLANE WITH NORMAL n

The shadow of the vector V onto the plane is obtained by subtracting the component that is off the plane, namely $\text{proj}_n(V)$

$$\Pi_{plane}(V) = V - \text{proj}_n(V)$$

assuming V has its tail on the plane. Its nice if we can put V at the origin, otherwise we have to pick a point r_o etc..., using $r_o = 0$ is preferred.

Functions of several variables

A function is an assignment of an output for each input. The set of all allowed inputs is the domain, the set of all outputs is the range. We will use the terminology

"f is an apple-valued function of oranges"

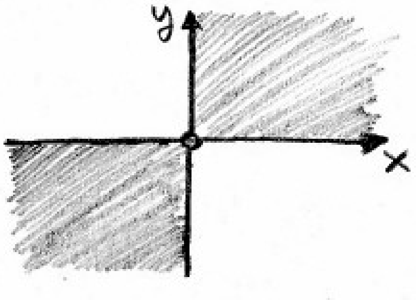
this means f takes in an orange and outputs an apple. Silly generalities aside we set apples $\subseteq \mathbb{R}$ for now and notice oranges $= \mathbb{R}$ (ma 241 and 141) or $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$ (ma 242). Notation is $f : \mathbb{R}^m \to \mathbb{R}$ however usually we have some $U \subset \mathbb{R}^m$ so $f : U \to \mathbb{R}$, then $\mathrm{dom}(f) = U$.

Def^n/ A function f of two variables is a rule that assigns each $(x,y) \in D \subseteq \mathbb{R}^2$ a unique real # $f(x,y)$.

$\mathrm{dom}(f) = D \qquad \mathrm{range}(f) = f(D) = \{ f(x,y) \mid (x,y) \in D \}$

All the same concerns as in precalc. enter here in the consideration of domains. Basically don't divide by zero and keep your square root inputs positive.

$\boxed{E18}$ $f(x,y) = \sqrt{xy}/(x^2+y^2)$ find $\mathrm{dom}(f)$. So we have to throw out the origin to avoid $\%$ by zero. Then we need $xy > 0 \Rightarrow$ either $x > 0$ and $y > 0$ OR $x < 0$ and $y < 0$.



the $\mathrm{dom}(f)$ consists of two disconnected parts.

Graphs of $f(x,y)$

to visualize $f(x)$ we need to look at $y = f(x)$ in $\mathbb{R}^2$.
to visualize $f(x,y)$ we need to look at $z = f(x,y)$ in $\mathbb{R}^3$.

$$\boxed{\mathrm{graph}(f(x,y)) \doteq \{ (x,y,z) \mid z = f(x,y),\ (x,y) \in \mathrm{dom}(f) \}}$$

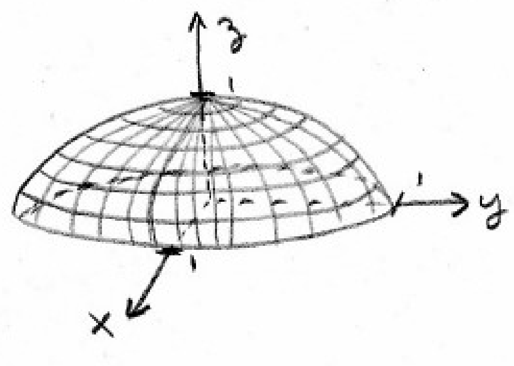
GRAPHS $z = f(x,y)$ and SURFACES in $\mathbb{R}^3$

Recall in $\mathbb{R}^2$ only shapes that passed the verticle line test could be interpreted as a graph $y = f(x)$, there were many other curves that were the collection of points in $\mathbb{R}^2$ satisfying some relation, usually algebraic. For example in $\mathbb{R}^2$ $x^2+y^2=1$ is a circle and it cannot be seen as $y=f(x)$ for a single function (it takes two in fact $\pm\sqrt{1-x^2}$). These distinctions persist in $\mathbb{R}^3$, surfaces of the form

$$z = f(x,y)$$

are quite special. We refer to them as a "graph" in $\mathbb{R}^3$. Stewart almost always finds a specialized formula for graphs, I'm much more interested that you understand the concept as opposed to using Stewart's cookbookish formulas. grr... I digress.

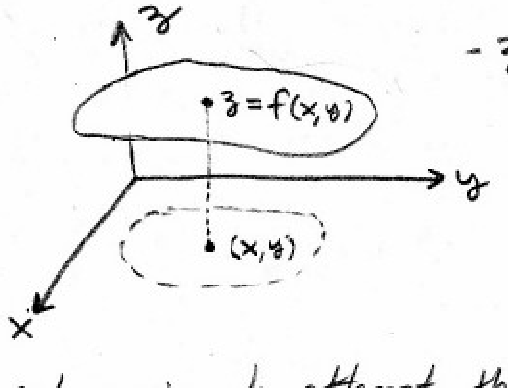
$\boxed{E19}$ $z = \sqrt{1-x^2-y^2}$ for $x^2+y^2 \leq 1$.



(this is supposed to be a hemisphere.)

(Use Maple!)

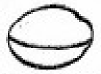
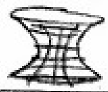
$\boxed{E20}$



$z = f(x,y)$

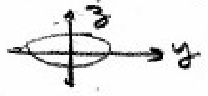
- I'm not going to attempt the pictures in §9.6. Please look over them to gain more intuition

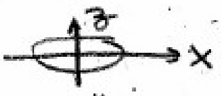
Quadric Surfaces : see table 2 of p. 682 for pretty pictures.

Ellipsoid :	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	 sphere if $a=b=c$
Elliptic Paraboloid :	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	
Hyperbolic Paraboloid :	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	
Cone :	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	
Hyperboloid of One Sheet :	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
Hyperboloid of Two Sheets :	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	

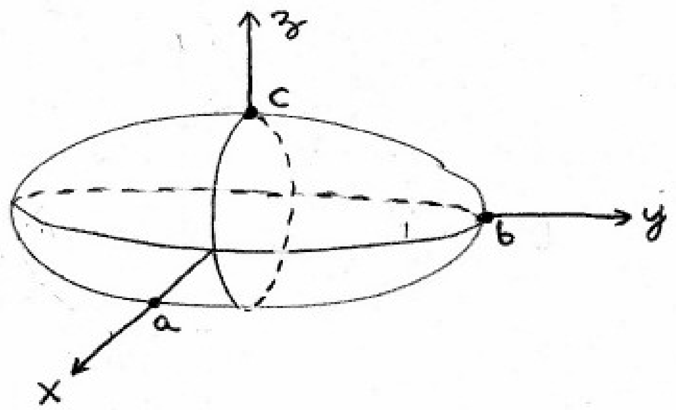
Lets study these as an exercise in how to graph by hand. (Assume Maple is evil, its the robot holocaust etc...)

$\boxed{\text{E21}}$ Ellipsoid. The name is appropriate, anyway we slice it we'll get an elliple. I usually look at the coordinate planes then go from there to gather whatever other data might appear to be helpful.

$x = 0 \Rightarrow y^2/b^2 + z^2/c^2 = 1$ 

$y = 0 \Rightarrow x^2/a^2 + z^2/c^2 = 1$ 

$z = 0 \Rightarrow x^2/a^2 + y^2/b^2 = 1$ 



Assuming $a, b, c > 0$ it looks something like this. Notice the idea is to use the cross-sections to get an idea where the surface is. This is essentially what maple does for us.

E22 Cone: $z^2 = x^2 + y^2$

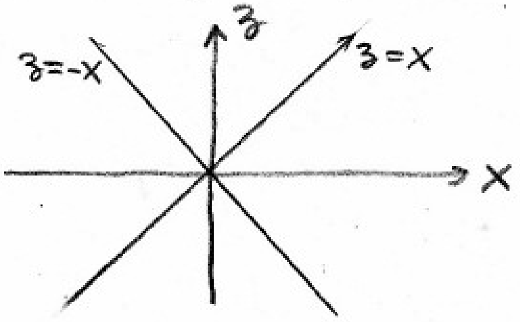
We can figure out somethings algebraically to begin,

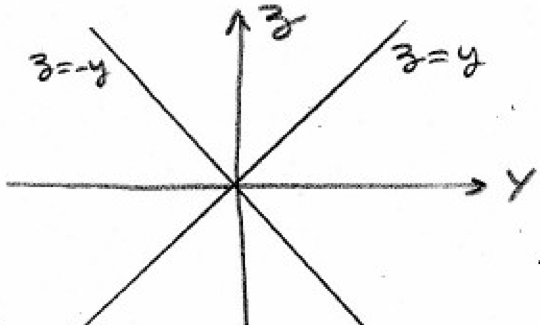
$$z = 0 = x^2 + y^2 \Rightarrow x = 0 \text{ and } y = 0 \therefore \text{ intersects } xy\text{-plane only at the origin.}$$

$$x = 0 \Rightarrow z^2 = y^2 \Rightarrow z = \pm y.$$

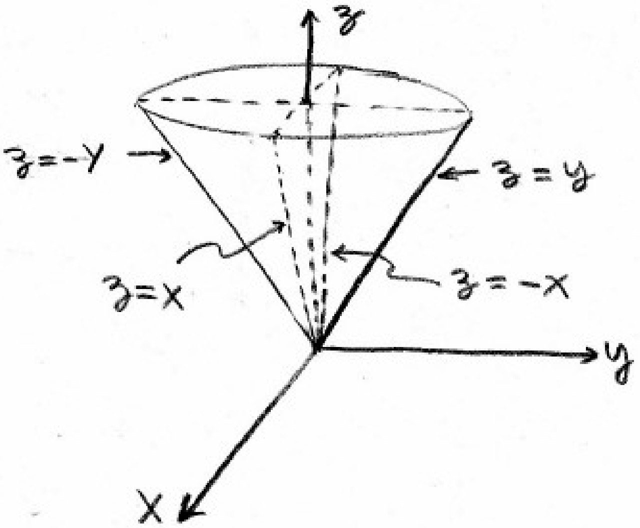
$$y = 0 \Rightarrow z^2 = x^2 \Rightarrow z = \pm x.$$

We can draw cross-sections of the surface with the coordinate planes $x=0$ and $y=0$





Then I'll attempt a 3-d rendition, (just for $z \geq 0$)



the trick is to draw the shape you imagine then check it with the cross-section lines (the text calls these "traces")

Remark: My point overall is that to plot a surface its helpful to look at particular slices then try to assemble the pieces. Certainly Maple is a big help here. I don't expect you to memorize the names of the surfaces (except the sphere I suppose thats common knowledge) but you should be able to find cross-sections by looking at coordinate planes and such. This may be important in setting up certain integrals later.

Functions of Three or more variables & Level Surfaces

Defⁿ/ A rule that assigns $(x,y,z) \in D$ to $f(x,y,z) \in \mathbb{R}$ is a function $f : D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^3$ is the domain of f, $\text{dom}(f) = D$. Also $\text{range}(f) = f(D) = \{f(x,y,z) \mid (x,y,z) \in D\}$.

- the definition for $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ should likewise be the obvious assignment of $(x_1, x_2, \dots, x_n) \longmapsto f(x_1, x_2, \dots, x_n) \in \mathbb{R}$.
- the $\text{graph}(f(x,y,z)) = \{(x,y,z,w) \mid w = f(x,y,z)\} \subseteq \mathbb{R}^4$ is kinda hard to visualize. We resort to level surfaces which are also helpful for visualizing $z = f(x,y)$ as it happens.

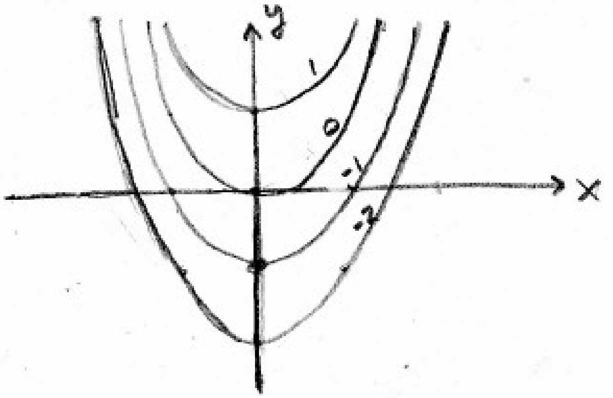
Defⁿ/ The level sets of $f : D \longrightarrow \mathbb{R}$ for $D \subseteq \mathbb{R}^n$ are the collections of points in D such that $f(D) = k$ where k is some constant in the range of f.

$n=2$) level curve for $f(x,y)$ is $f(x,y) = k$ (a curve)

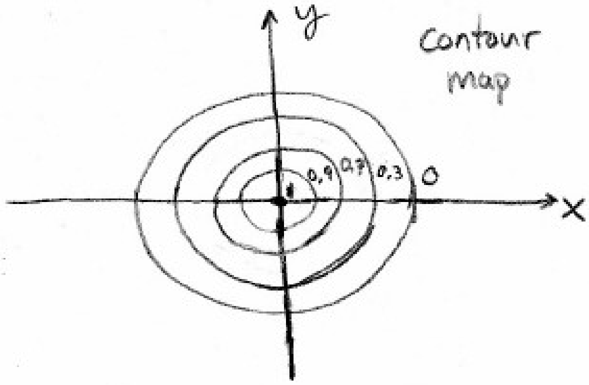
$n=3$) level surface for $f(x,y,z)$ is $f(x,y,z) = k$ (a surface)

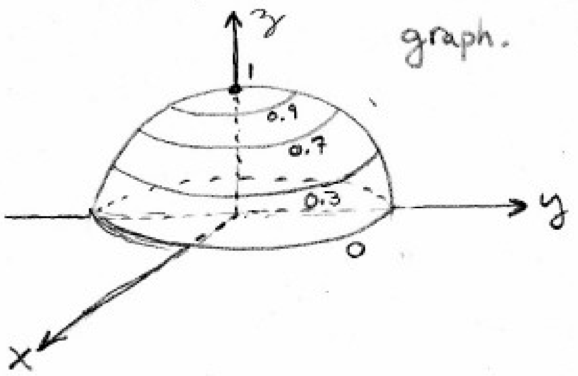
$n=4$) level volume for $f(x,y,z,w)$ is $f(x,y,z,w) = k$ (a volume!)

E23 Consider $z = y - x^2 = f(x,y)$ we can sketch a few level curves in xy-plane. This is a "contour map", you can imagine $z = f(x,y)$ using the contours. The values $1, 0, -1, -2$ indicate that $f(x,y) = 1, 0, -1, 2$ respectively along those curves. My artistic ability is insufficient to sketch this one. ☹



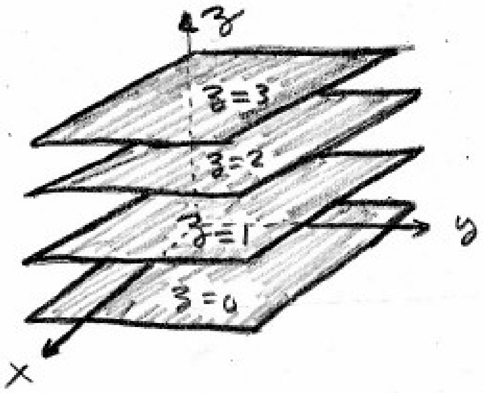
E24 Sketch a few contours for $z = \sqrt{1-x^2-y^2}$ then plot graph,





this crude sketch gives us a pretty good idea about what the graph $z = \sqrt{1-x^2-y^2}$ looks like.

E25 $f(x,y,z) = z$ plot several level surfaces. The level surfaces are



$$z = k$$

this is the plane at $z=k$ which is parallel to the xy-plane.

Remark: It takes some practice to get comfortable viewing surfaces. Certainly Maple is a good tool to help you and it would be wise to use it as an aid. We've seen how algebraic relations implicitly or explicitly define some curve or surface in $\mathbb{R}^3$. This is one description. The other description is the parametric one. You've seen that for curves, later we'll do parametrizations of surfaces. Both views are useful. Lastly I'd like to point out that cylindrical & spherical coordinates are a big conceptual aid in modelling surfaces, we delay discussion of that till later, closer to the time we'll use them most. If you wish to study them now and use them I wouldn't object.