Motion in Three Dimensions

We have all the mathematics we need, now we give the path $r : [a,b] \to C \subset \mathbb{R}^3$ where $t \longmapsto r(t) = \langle x(t), y(t), z(t) \rangle$ a physical interpretation. We suppose "t" is the time and consider a particular material object then for that object,

Def^n/
$$\boxed{\begin{aligned} \mathbf{r}(t) &\equiv \text{the position at time } t = \langle x(t), y(t), z(t) \rangle \\ V(t) &\equiv \frac{dr}{dt} \equiv \dot{r} \equiv \langle \dot{x}, \dot{y}, \dot{z} \rangle \equiv \text{velocity at time } t \\ a(t) &\equiv \frac{d^2r}{dt^2} \equiv \ddot{r} \equiv \langle \ddot{x}, \ddot{y}, \ddot{z} \rangle \equiv \text{acceleration at time } t \\ |V(t)| &\equiv |r'(t)| = \frac{ds}{dt} = \text{speed at time } t = \dot{s} \end{aligned}}$$

these are determined by Newton's Laws.. If F is the total force which acts on the object of mass m then Newton's 2nd law states that in terms of momentum $P \equiv mV$

$$\boxed{F = \frac{dP}{dt} = \frac{d}{dt}(mV) = \frac{dm}{dt}V + m\frac{dV}{dt}}$$

Now usually the mass m is constant so we have the familar

$$\boxed{F = ma}$$

E43 Ignoring orbital effects and for m near the surface of the earth the force of gravity is approximated by $F = m\langle 0, 0, -g \rangle$ where $g = 9.8\ m/s^2$. We take the xy-plane to be horizontal and z to be vertical. Further suppose initially $r(0) = \langle x_o, y_o, z_o \rangle = r_o$ and $V(0) = \langle 0, 0, V_o \rangle$.

$$F = m\langle 0,0,-g \rangle = m\langle \ddot{x}, \ddot{y}, \ddot{z} \rangle = m\langle \dot{V}_x, \dot{V}_y, \dot{V}_z \rangle = m\frac{dV}{dt}$$

$$\Rightarrow \int_0^t \frac{dV}{d\tau}\,d\tau = \int_0^t \langle 0, 0, -g \rangle\,d\tau$$

motion is purely vertical.

$$V(t) - V(0) = \langle 0, 0, -gt \rangle \quad \therefore \boxed{V(t) = \langle 0, 0, V_o - gt \rangle}$$

$$\frac{dr}{dt} = V(t) \Rightarrow \int_0^t \frac{dr}{d\tau}\,d\tau = \int_0^t \langle 0, 0, V_o - g\tau \rangle\,d\tau$$

$$r(t) - r(0) = \langle 0, 0, V_o t - \tfrac{g}{2}t^2 \rangle \quad \therefore \boxed{r(t) = \langle x_o, y_o, z_o + V_o t - \tfrac{g}{2}t^2 \rangle}$$

TANGENTIAL AND NORMAL COMPONENTS OF V and a

As we discussed the path instantaneously resides in the osculating plane. The velocity and acceleration are vectors in this plane and we can write V and a in terms of T and N. Also we attempt to further clarify the role of $\dot{s}$ and $\ddot{s}$ as they relate to the motion. As usual we assume the path is nonstop and nonlinear ($\dot{r} \neq 0$ and $\dot{r} \times \ddot{r} \neq 0 \ \forall t$)

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{V(t)}{ds/dt} \Rightarrow \boxed{V = \dot{s}\,T}$$

the velocity is purely tangential. Acceleration is not, consider,

$$a = \frac{d}{dt}(V) = \frac{d}{dt}(\dot{s}T) = \ddot{s}\,T + \dot{s}\frac{dT}{dt}$$

But we know that the derivative of T relates to N,

$$\frac{dT}{ds} = \kappa N \;\;\&\;\; \frac{dT}{dt} = \frac{ds}{dt}\frac{dT}{ds} \Rightarrow \frac{dT}{dt} = \dot{s}\kappa N$$

Therefore we find,

$$\boxed{a = \ddot{s}\,T + \dot{s}^2 \kappa N}$$

<u>Remark</u>: the magnitude of the acceleration is $|a| = \sqrt{(\ddot{s})^2 + (\dot{s}^2\kappa)^2}$. intuitively you might be tempted to suppose $|a| = \ddot{s} = \frac{d^2s}{dt^2}$ but this is not the case due to the Normal component. When $\kappa = 0$ (a case we must treat independently because the Frenet - Serret formulas apply to nonlinear nonstop paths) one has only the tangential component and $|a| = \ddot{s}$ (special case.)

<u>Components</u>:

To find the T or N components we simply take the dot-product with T or N (these are orthogonal $T \cdot N = 0$)

$$\boxed{\begin{aligned} a_T &\equiv a \cdot T = \ddot{s} \\ a_N &\equiv a \cdot N = \dot{s}^2 \kappa \end{aligned}}$$

CONSTANT SPEED CIRCULAR MOTION

Suppose that an object travels in a circle at at a constant angular velocity. In particular suppose after time T the object goes a full cycle around a circle of radius R; that is $V_o \equiv 2\pi R/T = \frac{ds}{dt}$. The parametrization fitting this description is the following if we assume $r(0) = (R, 0, 0)$.

$$r(t) = \langle R\cos(2\pi t/T),\ R\sin(2\pi t/T),\ 0\rangle$$

$$r'(t) = \frac{2\pi R}{T}\langle -\sin(\omega t),\ \cos(\omega t),\ 0\rangle \ \ : \text{letting } \omega \equiv \frac{2\pi}{T}$$

$$|r'(t)| = \frac{2\pi R}{T} = V_o = \omega R$$

$$T(t) = \langle -\sin(\omega t),\ \cos(\omega t),\ 0\rangle$$

$$T'(t) = \omega\langle -\cos(\omega t),\ -\sin(\omega t),\ 0\rangle$$

$$|T'(t)| = \omega$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \langle -\cos(\omega t),\ -\sin(\omega t),\ 0\rangle$$

Lets collect our thoughts,

$$V = V_o T$$

$$a = r''(t) = \omega^2 R\langle -\cos(\omega t),\ -\sin(\omega t)\rangle = \omega^2 R N$$

Compare this to our general result

$$a = \ddot{s}\,T + (\dot{s})^2 \kappa N = \omega^2 R N = \frac{V_o^2}{R^2} R N = \frac{V_o^2}{R} N$$

Therefore for <u>any</u> force that results in circular motion of a constant rate we must have

$$\boxed{F = ma = \frac{mV_o^2}{R} N = \frac{mV_o^2}{R}(-\hat{r})}$$

<u>Remark</u>: this is a very special case. Generally there is also $a_T \neq 0$. We note that to obtain circular motion of constant ω we need that $F \cdot T = 0$, probably the magnetic force $F = qV \times B$ is the most famous case. It turns out that even in the relativistic case it still causes circular motion. I'll show you if you ask.

$\boxed{E44}$ Let $r(t) = \langle t, 2t, t^2 \rangle$ find a_T and a_N. Calculate,

$$r'(t) = \langle 1, 2, 2t \rangle \Rightarrow \dot{s} = |r'(t)| = \sqrt{5+4t^2}$$

Now use our result from 281.

$$a_T = \ddot{s} = \frac{d}{dt}\left(\sqrt{5+4t^2}\right) = \boxed{\frac{4t}{\sqrt{5+4t^2}} = a_T}$$

To find a_N notice $|a|^2 = |a_T|^2 + |a_N|^2$ it's easy to calculate $a = r''(t) = \langle 0, 0, 2 \rangle$ thus $|a|^2 = 4$. So,

$$\begin{aligned} a_N &= \sqrt{|a|^2 - |a_T|^2} \\ &= \sqrt{4 - \frac{16t^2}{5+4t^2}} \\ &= \sqrt{\frac{20 + 16t^2 - 16t^2}{5+4t^2}} \\ &= \boxed{2\sqrt{5}/\sqrt{5+4t^2} = a_N} \end{aligned}$$

Remark: if we know κ then we could have use $a_N = \kappa \dot{s}^2$. On the other hand now we can find $\kappa = \frac{a_N}{\dot{s}^2}$

$\boxed{E45}$ Lets find a_T and a_N for the circle we studied explicitly on 282. We found that $\dot{s} = V_o$. Also we have calculated that $\kappa = 1/R$ for a circle of radius R in view of $\boxed{E37}$ on 275, notice we know the result holds despite the different parametrization on 282. The curvature is an <u>intrinsic</u> path independent property of the circle. Then using the results of 281,

$$a_T = \ddot{s}\,T = 0$$

$$a_N = \dot{s}^2 \kappa N = \frac{V_o^2}{R} N$$

$$\therefore \quad a = a_T T + a_N N = \boxed{\frac{V_o^2}{R} N = a}$$

<u>Remark</u>: there are many shortcut formulas. I don't require you to know them, if you use them then you should <u>derive</u> them on the test (not hwk though)

Kepler's Laws of Planetary Motion

In antiquity there have been radically different views of the universe at large and the motion or lack of motion of the earth through it. At the time of Kepler the heliocentric view of Copernicus (1473–1543) had taken hold, but astronomers insisted that planets traveled in circles, then circles on top of circles on top of circles... This system of "perfect" circles were known as epicycles. Epicycles worked quite well but Kepler (1571–1630) found them unatural. Kepler instead thought he could explain the motion of planets by a few simple rules. He found these rules emperically by studying the exquisite data taken by Tycho Brahe. These laws were chosen simply to fit the data. Only later were these laws derived from basic physical law. By the way, much of modern physics are still like Kepler's Laws, it is always the dream/goal/aspiration to derive known phenominological law from basic principles. There is some controversy as to who first derived Kepler's Laws, many credit Newton himself others credit Johann Bernoulli in 1710. The incredible thing is that we can derive the laws in a few short pages. Our notation and understanding of vector calculus is several hundred years in advance, so ordinary folks like myself can grasp the proof.

Set-up

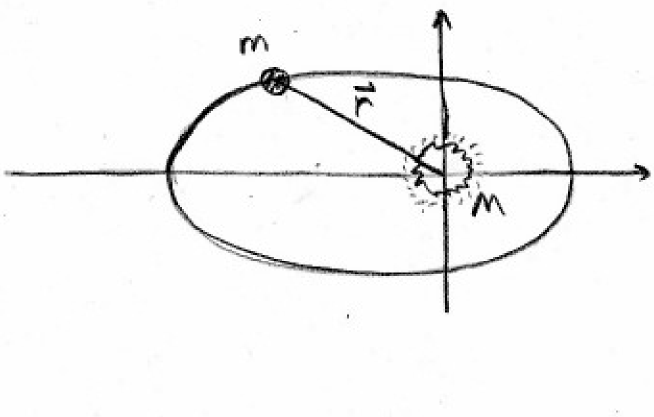
Keplers laws for the Sun and a single planet are:

1.) The orbit of the planet is elliptical with the sun at a focus.

2.) During equal times the planet sweeps out equal areas in the ellipse.

3.) $T^2 \propto a^3$ where T = period of planet's orbit, a = length of semimajor axis of ellipse.

We place the origin at the sun. We expect that



- My proof of Kepler's Laws follows Colley's of §3.1 fairly closely.

Proposition: The motion of the planet lies in a plane which also contains the sun if we assume Newton's Universal Law of Gravitation governs the motion through Newton's Laws.

Proof: our goal is to show that $\vec{r} \times \vec{V} = \vec{C}$ for some constant vector $\vec{C}$. This will show that planet moves in a plane with normal $\vec{C}$. Note,

$$\frac{d}{dt}(\vec{r} \times \vec{V}) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{V}}_{\vec{V} \times \vec{V} = 0.} + \vec{r} \times \frac{d\vec{V}}{dt} = \vec{r} \times \vec{a}.$$

Recall in our current notation that $\vec{r} = r\hat{r}$ and Newton tells us that,

$$\vec{F} = m\vec{a} = -\frac{GmM}{r^2}\hat{r} = -\frac{GmM}{r^3}\vec{r}$$

m = mass of planet
M = mass of sun
G = Gravitational Constant.

$$\therefore \quad \vec{a} = \frac{-GM}{r^3}\vec{r} \quad \text{thus } \vec{a} \parallel \vec{r}$$

$$\Rightarrow \vec{a} \times \vec{r} = 0 \quad \Rightarrow \quad \frac{d}{dt}(\vec{r} \times \vec{V}) = \vec{r} \times \vec{a} = 0 \quad \therefore \quad \underline{\vec{r} \times \vec{V} = \vec{C}}$$

Thm/ Kepler's 1st Law: The planet's orbit is an ellipse with sun at one focus

Proof: this will take a little work so be patient, lets get a better hold on $\vec{C}$,

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

Apply this to the following,

$$\vec{C} = \vec{r} \times \vec{V} = r\hat{r} \times [\cancel{\dot{r}\hat{r}}^{\,0} + r\frac{d\hat{r}}{dt}] = \underline{r^2\hat{r} \times \frac{d\hat{r}}{dt}} = \vec{C} \quad \text{(I.)}$$

Calculate then, using (I.)

$$\begin{aligned}
\vec{a} \times \vec{C} &= \left(\frac{-GM}{r^2}\hat{r}\right) \times \left(r^2\hat{r} \times \frac{d\hat{r}}{dt}\right) \\
&= -GM\left[\hat{r} \times \left(\hat{r} \times \frac{d\hat{r}}{dt}\right)\right] \\
&= GM\left[\left(\hat{r} \times \frac{d\hat{r}}{dt}\right) \times \hat{r}\right] \quad : \text{recall } \overbrace{A \times (B \times C) = (A \cdot C)B - (A \cdot B)C}^{\text{see §9.4 \#30}} \\
&= GM\left[(\hat{r} \cdot \hat{r})\frac{d\hat{r}}{dt} - \cancel{\left(\hat{r} \cdot \frac{d\hat{r}}{dt}\right)}^{\,0}\hat{r}\right] \quad : \hat{r} \cdot \hat{r} = 1 \Rightarrow 2\hat{r} \cdot \dot{\hat{r}} = 0 \Rightarrow \hat{r} \cdot \dot{\hat{r}} = 0. \\
&= GM\frac{d\hat{r}}{dt} \\
&= \underline{\frac{d}{dt}(GM\hat{r}) = \vec{a} \times \vec{C}} \quad \text{(II)}
\end{aligned}$$

Proof of Kepler's 1^{st} Law continued

We may derive another identity for $\vec{a}\times\vec{c}$,

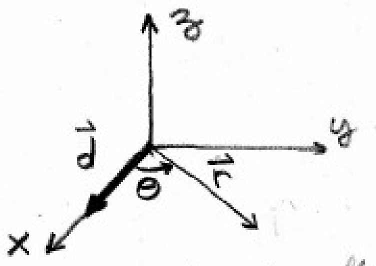
$$\vec{a}\times\vec{c} = \frac{d\vec{V}}{dt}\times\vec{c} + \vec{V}\times\frac{d\vec{c}}{dt} \quad : \text{added zero since } \frac{d\vec{c}}{dt} = 0.$$

$$= \underbrace{\frac{d}{dt}\left[\vec{V}\times\vec{c}\right]}_{III} \quad : \text{using identity (V.) on } 265$$

Thus comparing II. & III. we find

$$\frac{d}{dt}\left(GM\,\hat{r}\right) = \frac{d}{dt}\left(\vec{V}\times\vec{c}\right) \quad \therefore \quad \underbrace{\vec{V}\times\vec{c} = GM\hat{r} + \vec{d}}_{IV}$$

where $\vec{d}$ is a constant vector, it lies in the orbital plane since $\vec{v}\times\vec{c}$ and $\hat{r}$ do. Now choose coordinates in the orbital plane so that $\vec{d}$ lines up with the x-axis. Let θ be the usual θ in the xy-plane,



$$\hat{r}\cdot\vec{d} = |\hat{r}||\vec{d}|\cos\theta = d\cos\theta$$

where $|\vec{d}| = d$ in our notation here.

Now consider the length of $\vec{c}$ squared,

$$\begin{aligned}
c^2 &= \vec{c}\cdot\vec{c} \\
&= (\vec{r}\times\vec{V})\cdot\vec{c} \\
&= \vec{r}\cdot(\vec{V}\times\vec{c}) && : \text{using identity (V.) of } 248 \\
&= r\hat{r}\cdot\left[GM\hat{r} + \vec{d}\right] && : \text{using IV. we found just above.} \\
&= GMr + r\hat{r}\cdot\vec{d} \\
&= GMr + rd\cos\theta \\
&= r(GM + d\cos\theta)
\end{aligned}$$

Therefore we solve for $r = \sqrt{x^2+y^2+z^2} = \sqrt{x^2+y^2}$ (we're in $z=0$) and obtain the eq$^{=}$ of an ellipse (or parabola or hyperbola)

$$r = \frac{c^2}{GM + d\cos\theta} = \frac{c^2/GM}{1 + (d/GM)\cos\theta} = \boxed{\frac{p}{1 + e\cos\theta} = r}$$

where we define $p = c^2/GM$ and the eccentricity $e = d/GM$. This is an ellipse in polar coordinates. Since you've likely not seen that recently (or maybe never) we'll connect to the usual Cartesian eq^{n}'s for the ellipse. The details will be of use to us in proving the 3rd Law of Kepler later on.

$$r = \frac{p}{1+e\cos\theta} \Rightarrow r = p - er\cos\theta$$

Trying to convert the polar coordinates (r,θ) to (x,y) where $x = r\cos\theta$ and $y = r\sin\theta$. We see, using $x = r\cos\theta$

$$r = p - ex$$

$$r^2 = x^2 + y^2 = p^2 - 2epx + e^2x^2$$

$$x^2 - e^2x^2 + y^2 + 2epx = p^2$$

$$x^2(1-e^2) + 2epx + y^2 = p^2$$

$$x^2 + \frac{2ep}{1-e^2}x + \frac{y^2}{1-e^2} = \frac{p^2}{1-e^2} \quad : \quad \underline{\text{assume } e \neq \pm 1}$$

$$\left(x - \frac{ep}{1-e^2}\right)^2 + \frac{y^2}{(1-e^2)} = \frac{p^2}{1-e^2} + \frac{e^2p^2}{(1-e^2)^2} = \frac{p^2 - e^2p^2 + e^2p^2}{(1-e^2)^2} = \frac{p^2}{(1-e^2)^2}$$

$$\therefore \boxed{\frac{\left(x - \frac{ep}{1-e^2}\right)^2}{p^2/(1-e^2)^2} + \frac{y^2}{p^2/(1-e^2)} = 1} \quad \underset{(0<e<1)}{\underline{\text{ellipse}}} \ \text{OR} \ \underset{(e>1)}{\underline{\text{hyperbola}}}$$

This is an ellipse with center $(ep/(1-e^2), 0)$ and it has semimajor axis length $a = p/(1-e^2)$ and semiminor axis $b = p/\sqrt{1-e^2}$.
Remark: recall that we defined $p = c^2/GM$ so $p > 0$ and we need not worry about ÷ by p. Now $e = d/GM > 0$ so we can rule out $e = -1$ as a problem. Notice we have division by $\sqrt{1-e^2}$ as part of our sol^{n}, this only makes sense if $0 < e < 1$. The case $e = 1$ needs seperate treatment. Motion in the case $0 < e < 1$ is that of planets.

$\underline{e = 1}$ $\quad r = p - r\cos\theta \quad \therefore r^2 = (p-x)^2 = p^2 - 2xp + x^2$

that is $x^2 + y^2 = p^2 - 2xp + x^2 \Rightarrow 2xp = p^2 - y^2$

$$\therefore \boxed{x = p/2 - y^2/2p} \quad \underline{\text{parabola}}$$

Remark: One nice resource for background on conic-sections and polar coordinates is "Precalculus, Concepts through functions" Sullivan & Sullivan. There is just about all the cases you can imagine, rotated ellipses for example.

$\boxed{\text{Th}^{m}/ \text{ KEPLER'S } 2^{nd} \text{ LAW: During equal times a planet sweeps through equal areas.}}$

Proof: Pick a point P_o at angle θ_o. The later in this course we will learn that the area in polar coordinates swept by the region from θ_o to θ is simply

$$A(\theta) = \int_{\theta_o}^{\theta} \frac{1}{2} r^2 \, d\beta$$

We seek to show that $\frac{dA}{dt} =$ constant. Consider then

$$\frac{dA}{d\theta} = \frac{d}{d\theta} \int_{\theta_o}^{\theta} \frac{1}{2} r^2 \, d\beta = \frac{1}{2} r^2 \qquad \text{by F.T.C.}$$

Then the chain rule tells us

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

Notice that $\hat{r} = \langle \cos\theta, \sin\theta \rangle$ thus diff. implicitly, remember $\theta = \theta(t)$.

$$\frac{d\hat{r}}{dt} = \langle -\sin\theta, \cos\theta \rangle \frac{d\theta}{dt} = \langle -\sin\theta, \cos\theta, 0 \rangle \frac{d\theta}{dt} \quad \text{(we've been suppressing the z-comp.)}$$

(285), (I) $\Rightarrow$ $\vec{C} = r^2\left(\hat{r} \times \frac{d\hat{r}}{dt}\right) = r^2 \frac{d\theta}{dt} \langle \cos\theta, \sin\theta, 0 \rangle \times \langle -\sin\theta, \cos\theta, 0 \rangle$

$$\vec{C} = r^2 \frac{d\theta}{dt} \langle 0, 0, 1 \rangle \quad \therefore \quad C = r^2 \frac{d\theta}{dt}$$

Hence $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{C}{2} =$ constant. //

$\boxed{\text{Th}^{m}/ \text{ KEPLER'S } 3^{rd} \text{ Law: } T^2 = Ka^3 \text{ where } T \text{ is the orbital period and } a \text{ is the length of the semimajor axis, } K = \text{some constant}}$

Proof: I proved back on pg. (138) in $\boxed{\text{E7}}$ that the area of an ellipse is $A = \pi ab$. On the other hand we could say that $dA = \frac{dA}{dt} dt$ and integrate over a whole orbit to find

$$\pi ab = \int_0^T \frac{dA}{dt} dt = \int_0^T \frac{C}{2} dt = \frac{CT}{2} \quad \therefore \quad T = \frac{2\pi ab}{C} \quad \therefore \quad T^2 = \frac{4\pi^2 a^2 b^2}{C^2}$$

notice that $a^2 = p^2/(1-e^2)^2$ and $b^2 = p^2/(1-e^2)$, also $C^2 = GMp$.

$$T^2 = \frac{4\pi^2}{GMp} \frac{p^2}{(1-e^2)^2} \cdot \frac{p^2}{(1-e^2)} = \frac{4\pi^2}{GM}\left(\frac{p}{1-e^2}\right)^3 = \boxed{\frac{4\pi^2 a^3}{GM} = T^2} \; //.$$

It is interesting that $K = \frac{4\pi^2}{GM}$ is independent of the planets mass. all the planets orbit under the same K-value.

Remark: I have spent some effort presenting Kepler's Laws. I may ask you to prove some subset, I'll pin it down in the test review sheet.

Remark: There is another method of proving Kepler's Laws that begins with the two-body Lagrangian for a central potential (well force really but $F = f(r)\hat{r} \Rightarrow U = U(r) \ldots$). In that derivation one need not assume the sun is at the origin. Instead you consider the center of mass to be at the origin and work out how the reduced mass μ orbits. Anyway its very beautiful, take PY 411 if you wish to see the more general derivation. Also they will actually find $r(t)$ explicitly as opposed to the indirect arguments we have offered (or rather stolen from Colley ☺.)