(We expand on §11.4 of Stewart in essence.)
To begin we consider $z=f(x,y)$, we assume that f_x and f_y are continuous so we are assured that the tangent plane is well defined (see 306 for what happens otherwise).

PROPOSITION: The tangent plane to $z=f(x,y)$ at $(a,b,f(a,b))$ is

$$z = f(a,b) + \frac{\partial f}{\partial x}\Big|_{(a,b)}(x-a) + \frac{\partial f}{\partial y}\Big|_{(a,b)}(y-b)$$

provided that f is differentiable at (a,b).

Pf/. will have to wait until we give a better technical description of what a tangent plane is theoretically. Ignorance of that def$^{\underline{n}}$ will not hinder us in our work on graphs. You may skip ahead to 318-319 for details, or see 306 for the def$^{\underline{n}}$.

Def$^{\underline{n}}$/ Let $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (a,b). We say $L: \mathbb{R}^2 \to \mathbb{R}$ is the linearization of f at (a,b) defined by

$$\boxed{L(x,y) = f(a,b) + \frac{\partial f}{\partial x}\Big|_{(a,b)}(x-a) + \frac{\partial f}{\partial y}\Big|_{(a,b)}(y-b)}$$

we may add an f and (a,b) if we have several linearizations and need a distinguishing notation ($L = L^f_{(a,b)}$ or $L_{f(a,b)}$ perhaps)

__Remark:__ the linearization L of f is the __best linear approximation__ of the function near the base point of the linearization. This is the natural generalization of the tangent line approx. of a function, the closer to the point of tangency the closer the tangent line approximates the function.

E71 Find the eq$^{\underline{n}}$ of the tangent plane at $(1,2,5)$ for $f(x,y) = x^2+y^2$. We calculate $f_x(1,2)$ and $f_y(1,2)$,

$$f_x(1,2) = 2x\big|_{(1,2)} = 2 \qquad f(1,2) = 1^2+2^2 = 5$$

$$f_y(1,2) = 2y\big|_{(1,2)} = 4$$

Thus the tangent plane is $\boxed{z = 5 + 2(x-1) + 4(y-2)}$

TANGENT PLANES, LINEARIZATIONS, TOTAL DERIVATIVE

$\boxed{E72}$ Find the linearization of $f(x,y)=x^2+y^2$ at $(1,2)$. Then approximate $f(2,2)$ and compare to the real-value. We found the tangent plane's eq$^{\underline{n}}$ in $\boxed{E71}$ so we already know

$$\boxed{L(x,y) = 5+2(x-1)+4(y-2)}$$

We approximate f via L,

$$f(2,2) \cong L(2,2) = 5+2(2-1)+4(0) = 7$$

Of course we can just evaluate $f(2,2) = 2^2+2^2 = 8$ to see we have an absolute error of $8-7=1$. We can express these thoughts via the "increment" and "total differential"

$$\Delta z = f(x_2,y_2) - f(x_1,y_1)$$

total differential $\rightarrow$ $dz = f_x(x_1,y_1)\underbrace{(x_2-x_1)}_{dx} + f_y(x_1,y_1)\underbrace{(y_2-y_1)}_{dy}$

In particular we have $(x_1,y_1)=(1,2)$ and $(x_2,y_2)=(2,2)$ thus $dx=1$ and $dy=0$ hence

$$dz = 2dx+4dy = 2+0 = 2$$
$$\Delta z = f(2,2)-f(1,2) = 8-5 = 3$$

Then $f(2,2) = f(1,2)+\Delta z = 5+3 = 8$ (the true value)
$L(2,2) = f(1,2)+dz = 5+2 = 7$ (the approximate value)

Remark: this is a reasonable notation for approximation work, but I much prefer to use $\Delta x = x_2-x_1$ and $\Delta y = y_2-y_1$ for finite increments. Conceptually, when I write dx or dy then I have in mind an *infinitesimal* change in x or y. Stewart does not share my vision, see E4 on p. 775.

$$\boxed{\text{Def}^{\underline{n}}/ \text{ The total differential of } z=f(x,y) \text{ is defined to be } \quad dz \equiv \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \equiv df}$$

- notice this is for $f(x,y)$. When we have $w=f(x,y,z)$ then we will write $dw = \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz = \frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz$

(since $w=f$)

[E73] It is known that if we place resistors R_1 and R_2 in parallell then the effective resistance R of the system is

$$R = \frac{1}{R_1} + \frac{1}{R_2}$$

we can view $R = f(R_1, R_2)$, it is a function of two variables. Now suppose $R_1 = 10\Omega \pm 1\Omega$ and $R_2 = 2\Omega \pm 0.5\Omega$, here Ω = "ohm" and we'll drop them for convenience sake. Interpretation:

$$dR_1 = 1 \qquad \text{and} \qquad dR_2 = 0.5$$

what is the order of our uncertainty in R then? Essentially upto a convention or two its the total differential in R,

$$\begin{aligned}
dR &= \frac{\partial R}{\partial R_1}\bigg|_{(10,2)} dR_1 + \frac{\partial R}{\partial R_2}\bigg|_{(10,2)} dR_2 \\
&= \left(\frac{-1}{R_1^2}(1) - \frac{1}{R_2^2}(0.5)\right)\bigg|_{(10,2)} \\
&= \left(\frac{-1}{100} - \frac{0.5}{4}\right) \\
&= \frac{-1}{100} - \frac{1}{8} = \boxed{-0.115 = dR} \quad (\text{Mostly from } \Delta R_2.)
\end{aligned}$$

Remark: this is the largest uncertainty or error if you prefer. Be warned you should study error & measurement & statistics elsewhere.

[E74] Let $R = R_1 + R_2 = g(R_1, R_2)$. That is assume the resistors are in series this time. Find dR in this case,

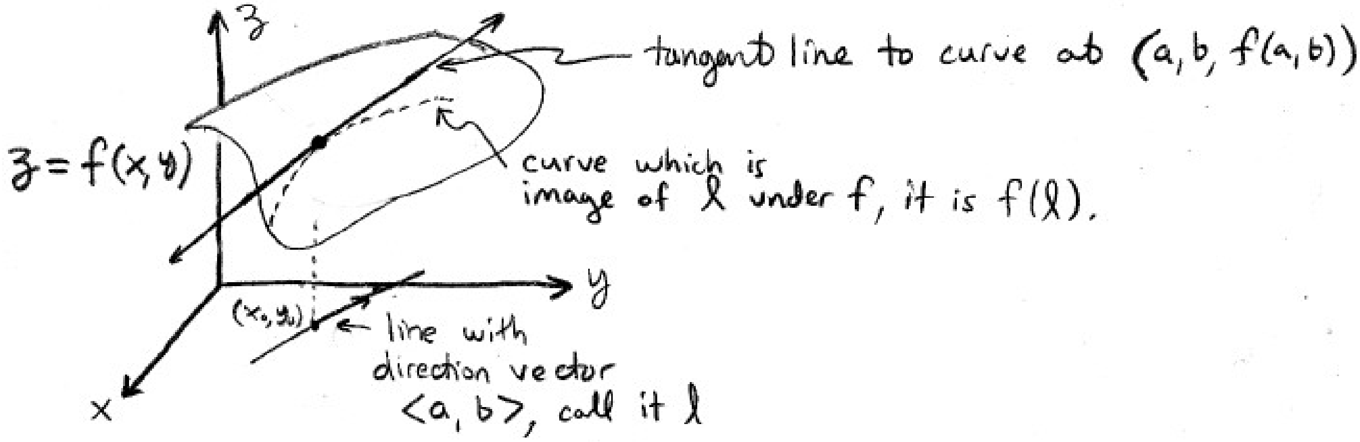
$$\begin{aligned}
dR &= \frac{\partial R}{\partial R_1}\bigg|_{(10,2)} dR_1 + \frac{\partial R}{\partial R_2}\bigg|_{(10,2)} dR_2 \\
&= dR_1 + dR_2 \\
&= 1 + 0.5 \\
&= \boxed{1.5 = dR} \quad (\text{Mostly from } \Delta R_1)
\end{aligned}$$

Remark: You can see the net-error is a consequence of both the error in the inputs and the eq^{n} that gives the output. The total differential gives us an estimation of that net-error.

The directional derivative of f at (x_o, y_o) in the direction $\hat{u} = (a, b)$ is the rate of change in f in that direction from (x_o, y_o),

$$\boxed{D_{\hat{u}} f(x_o, y_o) \equiv \lim_{h \to 0} \frac{f(x_o + ah, y_o + bh) - f(x_o, y_o)}{h}}$$

Here the notation $D_u f(x_o, y_o)$ is usually employed, but take note that $|u| = 1$ is required. The geometry of this is



in the special cases $\langle a, b \rangle = \langle 1, 0 \rangle$ or $\langle 0, 1 \rangle$ we obtain plain-old partial derivatives, its the same idea of tangency to $z = f(x, y)$ just tilted. That is,

$$D_{\langle 1,0 \rangle} f(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \equiv f_x(a, b).$$

$$D_{\langle 0,1 \rangle} f(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h} \equiv f_y(a, b).$$

$$\boxed{\underline{\text{Proposition:}}\ D_{(a,b)} f(x_o, y_o) = f_x(x_o, y_o)\, a + f_y(x_o, y_o)\, b}$$

<u>Proof:</u> see p. 791, its an easy consequence of the chain rule.

> **Def^n/ for** The gradient of f is denoted grad(f) or ∇f and $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ it is defined by
>
> $$\text{grad}(f) = \nabla f = \langle f_x, f_y \rangle$$
>
> Or at a point. $(\nabla f)(x_o, y_o) = \langle f_x(x_o, y_o), f_y(x_o, y_o) \rangle$.

<u>Observation:</u> $\underline{D_{\langle a,b \rangle} f(x_o, y_o) = (\nabla f)(x_o, y_o) \cdot \langle a, b \rangle}$

this is a nice way to remember it.

Examples of ∇f and directional derivatives

$\boxed{E75}$ Let $f(x,y) = \ln(xy)$. Find the rate of change in f at the point $(1,2)$ in the $\langle 1,-1\rangle$ direction. To begin we calculate the gradient,

$$\nabla f = \left\langle \frac{\partial}{\partial x}[\ln(xy)], \frac{\partial}{\partial y}[\ln(xy)] \right\rangle = \left\langle \frac{1}{x}, \frac{1}{y} \right\rangle$$

notice $\langle 1,-1\rangle$ has length $\sqrt{1^2+1^2} = \sqrt{2}$ so the unit vector we need here is $u = \frac{1}{\sqrt{2}}\langle 1,-1\rangle$. So calculate,

$$D_u f(1,2) = \left\langle \tfrac{1}{1}, \tfrac{1}{2} \right\rangle \cdot \left\langle 1/\sqrt{2}, -1/\sqrt{2} \right\rangle$$
$$= 1/\sqrt{2} - \tfrac{1}{2}(1/\sqrt{2}) = \tfrac{1}{\sqrt{2}}\left(1-\tfrac{1}{2}\right) = \frac{1}{2\sqrt{2}} = \boxed{\frac{\sqrt{2}}{4}}$$

in geometric terms this says the slope of the curve on $z = \ln(xy)$ above $r(t) = \langle 1,2\rangle + t\langle 1,-1\rangle$ at $(1,2,\ln(2))$ has slope $\sqrt{2}/4$. This is how quickly f changes in the $\langle 1,-1\rangle$ direction at $(1,2)$.

- <u>What $\langle a,b\rangle$ give min/max rates of change at $(1,2)$?</u>

Recall that $\vec{V}\cdot\vec{W} = |\vec{V}||\vec{W}|\cos\theta$ and $-1 \le \cos\theta \le 1$ so we obtain max. when $\theta = 0$ and min. when $\theta = \pi$.

$$\Delta f_{max} = |\nabla f(1,2)| = \sqrt{1^2 + \left(\tfrac{1}{2}\right)^2} = \sqrt{\frac{4+1}{4}} = \boxed{\frac{\sqrt{5}}{2}} \text{ max rate.}$$

$$\Delta f_{min} = -|\nabla f(1,2)| = -\sqrt{5}/2.$$

We find that $\langle 1/\sqrt{5}, -1/\sqrt{2}\rangle$ gives max rate of change of f at $(1,2)$ while $\langle -1/\sqrt{2}, 1/\sqrt{2}\rangle$ gives minimum rate of change of f at $(1,2)$.

$\boxed{E76}$ Let $f(x,y) = x\sin(y) + y\cos(x)$ find rate of change in the $\langle 1,1\rangle$ direction at (π,π). Note $u = \frac{1}{\sqrt{2}}\langle 1,1\rangle$,

$$D_u f(\pi,\pi) = \left\langle \sin(y) - y\sin(x), x\cos y + \cos(x) \right\rangle\Big|_{(\pi,\pi)} \cdot \left(\tfrac{1}{\sqrt{2}}\langle 1,1\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left[(\sin\pi - \pi\sin\pi)1 + (\pi\cos\pi + \cos\pi)1\right]$$
$$= \boxed{\frac{-1}{\sqrt{2}}(\pi+1)}$$

SUMMARY: $D_u f(x_0,y_0)$ gives the rate of change of f at (x_0,y_0) in the u-direction. If $u = \langle a, b\rangle$ then

$$D_u f(x_0,y_0) = \nabla f(x_0,y_0)\cdot u = f_x(x_0,y_0)\,a + f_y(x_0,y_0)\,b$$

The max/min rates of change at (x_0,y_0) occur in the $\pm\nabla f(x_0,y_0)$ directions, with values $\pm|\nabla f(x_0,y_0)|$.

DIRECTIONAL DERIVATIVES OF FUNCTIONS OF THREE OR MORE VARIABLES

The gradient vector for $f: X \subseteq \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\boxed{\nabla f \equiv \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}\right\rangle}$$

Of course we usually encounter $n=2$ or 3, $n=2$ we've discussed and used many times, $n=3$ is basically the same

$$\boxed{\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \quad \text{where } f = f(x,y,z)}$$

The rate of change of $f: X \subseteq \mathbb{R}^3 \to \mathbb{R}$ at P in the $\langle a,b,c\rangle$ direction is simply a a,b,c weighted sum of the change of f at P in the x,y,z-directions, we call this the DIRECTIONAL DERIVATIVE of f in the $\langle a,b,c\rangle$ direction at the point $P=(P_1,P_2,P_3)$

again assumed to have length one.

$$\boxed{D_{\langle a,b,c\rangle} f(P) \equiv (\nabla f)(P_1,P_2,P_3)\cdot\langle a,b,c\rangle}$$

again it should be emphasized $|\langle a,b,c\rangle| = 1$ is assumed here.

$\boxed{\text{E77}}$ Let $f(x,y,z) = x^2+y^2+z^2$. Find the maximum rate of change at the point $(1,1,1)$. To begin find ∇f

$$\nabla f = \langle 2x, 2y, 2z\rangle$$

$$\nabla f(1,1,1) = \langle 2,2,2\rangle$$

Then $D_u f(1,1,1) = \langle 2,2,2\rangle\cdot u$, this is maximized when $u \parallel \nabla f$. We need to find unit vector, so divide by the length of $\langle 2,2,2\rangle$ to obtain

$$u = \frac{1}{\sqrt{12}}\langle 2,2,2\rangle = \boxed{\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\rangle = u}$$

- we see in higher dimensions the directional derivative works the same, I will not try to graph this situation as it is in 4-dimensional space.

PARAMETRIZED SURFACES

We have seen that surfaces in $\mathbb{R}^3$ can be described as level surfaces ($x^2+y^2+z^2=1$ for example) or possibly as the graph of some function of two variables ($z = x+y$ for example). The case of the graph is really just a special case of the level surface idea, given $z = f(x,y)$ we can always construct $F(x,y,z) = z - f(x,y)$ and describe $z = f(x,y) \Leftrightarrow F(x,y,z) = 0$.

There is another approach to describing a surface in $\mathbb{R}^3$.

Def^n/ $\Phi : D \subseteq \mathbb{R}^2 \longrightarrow S \subseteq \mathbb{R}^3$ is a parametrization of the surface S. We say D is the parameter space and denote

$$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

the mapping Φ should be smooth

E78 The sphere $x^2+y^2+z^2 = 1$ has a natural parametrization in terms of the polar angle θ and azimuthal angle φ,

$$x = \cos\theta \sin\varphi \qquad y = \sin\theta\sin\varphi \qquad z = \cos\varphi$$

Notice that

$$\begin{aligned} x^2+y^2+z^2 &= \cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\varphi \\ &= \sin^2(\cos^2\theta + \sin^2\theta) + \cos^2\varphi \\ &= \sin^2\varphi + \cos^2\varphi \\ &= 1 \qquad \text{(this shows the parametrization works)} \end{aligned}$$

Thus $\Phi(\theta,\varphi) = (\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi)$ parametrizes the sphere, we take $D = [0,2\pi]\times[0,\pi]$ in this case since we want our angles to range $0 \le \theta \le 2\pi$ and $0 \le \theta \le \pi$.

E79 Let $z = f(x,y)$ be a graph of some smooth function $f : D \to \mathbb{R}^2$ then we can choose x & y as our parameters and write

$$\Phi(x,y) = (x,\ y,\ f(x,y))$$

thus we find graphs are a special case of parametrically described surfaces.

E80 Consider $x^2+y^2-z^2=1$ this is a hyperboloid. If you know a little about hyperbolic trig. functions then the following parametrization is fairly obvious

$$x = \cosh\gamma \cos\theta \qquad y = \cosh\gamma \sin\theta \qquad z = \sinh\gamma$$

$$\begin{aligned} x^2+y^2-z^2 &= \cosh^2\gamma\cos^2\theta + \cosh^2\gamma\sin^2\theta - \sinh^2\gamma \\ &= \cosh^2\gamma - \sinh^2\gamma \\ &= 1. \quad \text{(it works)} \end{aligned}$$

We'd say γ is a hyperbolic angle, in contrast to the ordinary angle θ we must allow γ to range over $(-\infty,\infty)$ whereas $0 \le \theta \le 2\pi$. The mapping in total is $X(\gamma,\theta) = (\cosh\gamma\cos\theta, \cosh\gamma\sin\theta, \sinh\gamma)$

Remark: you've probably noticed that we don't have to use "u" and "v" for the parameters. Likewise the mapping X is often replaced with $\mathbf{r}$ in the text or my homework sol^{0}'s.

TANGENT PLANES:

We give the methodology for each description of a surface

I.) If $F(x,y,z)=0$ and (x_o,y_o,z_o) is a point on that surface then the tangent plane at (x_o,y_o,z_o) is

$$(\nabla F)(x_o,y_o,z_o)\cdot(x-x_o,\, y-y_o,\, z-z_o) = 0.$$

II.) If $z=f(x,y)$ and $z_o = f(x_o,y_o)$ then the tangent plane at (x_o,y_o,z_o),

$$z = z_o + f_x(x_o,y_o)(x-x_o) + f_y(x_o,y_o)(y-y_o).$$

this is really just a special case of I.) just take $F = f(x,y) - z$.

Tangent Planes Continued:

III.) Given a parametrized surface $\Sigma(u,v)$ we study a particular point $\Sigma(u_o, v_o)$. Notice we have two curves through the point which are constructed by fixing one of the parameters and letting the other vary, lets denote them α & β,

$$\alpha(u) \equiv \Sigma(u, v_o)$$
$$\beta(v) \equiv \Sigma(u_o, v)$$

The tangent vectors to these curves are given by the partial derivatives of the map Σ,

$$\alpha'(u) = \frac{\partial \Sigma}{\partial u}(u, v_o) \quad \& \quad \beta'(v) = \frac{\partial \Sigma}{\partial v}(u_o, v)$$

then at the point of interest we have two tangent vectors to the surface and presumably $\alpha'(u_o) \times \beta'(v_o) \neq 0$, actually this is not universally true for all surfaces, but it is true for <u>oriented</u> surfaces where $\Sigma_u \times \Sigma_v \neq 0 \quad \forall u, v$.

$$N(u_o, v_o) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial u & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \end{vmatrix}\Bigg|_{(u_o, v_o)}$$

$$= \left\langle \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \right\rangle \Bigg|_{(u_o, v_o)}$$

then we have a normal vector $N(u_o, v_o)$ and a point so we construct the plane in the usual way

$$N(u_o, v_o) \cdot (\vec{r} - \Sigma(u_o, v_o)) = 0 \quad , \quad \vec{r} = \langle x, y, z \rangle.$$

<u>Remark</u>: its very simple, we get the normal one of two ways

(i) by a gradient ∇F normal to $F = 0$

(ii.) by cross product of tangents $N = \Sigma_u \times \Sigma_v$ to $\Sigma(u, v)$.