Maximizing & Minimizing Functions of Several Variables

We generalize the terminology of calc. I to fit our purposes here, the meaning of local min/max and absolute min/max have the obvious meanings.

Def^{n}/ $f : \text{dom}(f) \subseteq \mathbb{R}^2 \to \mathbb{R}$ has a local maximum at (a,b) if $f(x,y) \leq f(a,b)$ when (x,y) is on some disk centered at (a,b). Likewise $f(a,b)$ is a local minimum of f if $f(x,y) \geq f(a,b)$ when (x,y) is on some disk centered at (a,b). If we have that $f(x,y) \leq f(a,b)$ $\forall (x,y) \in S \subseteq \text{dom}(f)$ then we say that $f(a,b)$ is the maximum of f on S. Likewise if $f(x,y) \geq f(a,b)$ $\forall (x,y) \in S \subseteq \text{dom}(f)$ then $f(a,b)$ is the minimum of f on S. When $S = \text{dom}(f)$ we call those a global maximum or minimum.

let me list the theoretical tools then we'll do a few examples.

Th^{m}/ If f has a local max/min at (a,b) and the first-order partial derivatives exist there then $(\nabla f)(a,b) = \langle f_x(a,b), f_y(a,b)\rangle = 0$.

Def^{n}/ say (a,b) is a critical point if either $(\nabla f)(a,b) = 0$ or one or both of the partial derivatives do not exist.

as the theorem indicates if we have a local max/min at which the partials exist then that point must be a critical point. We cannot reverse this though, just because (a,b) is a critical point that does not gaurantee that $f(a,b)$ is a max/min. Just as in the $y = f(x)$ case we'll want to check all the critical points to see if they're extremal.

Th^{m}/ Suppose the 2nd partial derivatives of f are continuous on a disk centered on (a,b) and $(\nabla f)(a,b) = 0$. Define

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

then we have three cases

(i.) If $D > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is local min.

(ii) If $D > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is local max.

(iii.) If $D < 0$, then $f(a,b)$ is not a local max/min. We say that $f(a,b)$ is a <u>saddle point</u> of f in this case.

Examples:

$\boxed{E81}$ Find local max/min and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

$$\nabla f = \langle 4x^3 - 4y,\ 4y^3 - 4x \rangle = 0 \quad \text{at critical points.}$$

$$\nabla f = 0 \quad \begin{array}{l} \nearrow 4x^3 - 4y = 0 \rightarrow y = x^3 \\ \searrow 4y^3 - 4x = 0 \rightarrow x = y^3 \end{array} \Big\rangle \rightarrow x = x^9$$

Thus $x^9 - x = x(x^8-1) = x(x^4+1)(x^4-1) = \underbrace{x(x^4+1)(x^2+1)(x^2-1)}_{\text{give real roots } x=0,\pm 1.} = 0$

We find critical points $(0,0)$, $(1,1)$, $(-1,1)$. Now apply the 2nd derivative test Th^m, but first find D,

$$D(x,y) = f_{xx}f_{yy} - [f_{xy}]^2 = (12x^2)(12y^2) - [-4]^2 = 144x^2y^2 - 16.$$

we organize our results in a table

Critical Point	Value of f	f_{xx}	D	Conclusion
$(0,0)$	1	0	$-16<0$	saddle point
$(1,1)$	-1	12	$128>0$	local min.
$(-1,-1)$	-1	12	$128>0$	local min.

$\boxed{E82}$ find the shortest distance from $(1,0,-2)$ to the plane $x+2y+z=4$.

Define $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$ the distance from $(1,0,-2)$ to (x,y,z) then if on the plane we have $z = 4 - x - 2y$ so

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$

Now minimize $f(x,y) = d^2 = (x-1)^2 + y^2 + (6-x-2y)^2$

$$f_x = 2(x-1) - 2(6-x-2y) = 4x - 14 + 4y$$
$$f_y = 2y - 4(6-x-2y) = 10y - 24 + 4x$$

Critical points have $f_x = 0$ and $f_y = 0$ so, (f_x, f_y exist continuously everywhere)

E82 continued: We need $f_x = 0$ and $f_y = 0$ for critical point, this amounts to two eq^{n}'s & two unknowns here,

$$\begin{array}{r} (4x + 10y = 24) \\ -(4x + 4y = 14) \\ \hline 6y = 10 \quad \therefore \quad y = 10/6 = 5/3 \end{array}$$

then $x = 6 - \frac{10y}{4} = 6 - \frac{10}{4}\frac{10}{6} = 6 - \frac{100}{24} = 6 - \frac{25}{6} = \frac{36-25}{6} = \frac{11}{6}$

the critical point is $(11/6, 5/3)$. Now find D,

$$D = f_{xx}f_{yy} - [f_{xy}]^2 = (4)(10) - (0) = 40 > 0 \quad \text{and } f_{xx} = 4 > 0$$

thus we have a local minimum. So the closest point is where $x = 11/6$, $y = 5/3$ and $z = 4 - \frac{11}{6} - \frac{10}{3} = \frac{96-44-80}{24} = \frac{-28}{24} = \frac{-7}{6}$

that is $(11/6, 5/3, 7/6)$ is the closest point on the plane $z = 4 - x - 2y$ to the point $(1, 0, -2)$. The distance is

$$d = \sqrt{\left(\frac{11}{6} - 1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{-7}{6} + 2\right)^2} = \sqrt{\frac{25 + 100 + 25}{36}} = \sqrt{\frac{6(25)}{36}} = \boxed{\frac{5\sqrt{6}}{6}}$$

Remark: You might wonder how do I maximize a function of three or more variables? The answer is not found in Stewart or Thomas for that matter. Colley has the answer, take a look at §4.2 p. 251 the "Second derivative Test for local extrema" We use $D = f_{xx}f_{yy} - [f_{xy}]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$ the test for functions of three variables is based of the "Hessian" of the function,

$$Hf = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

take a look in Colley for the details, its not to tricky.

Absolute Maximums and Minimums

In calc I. we found a procedure for locating the max/min of $f(x)$ on a closed interval $[a,b]$. We now discuss the generalization of that to $f(x,y)$ for closed and bounded subsets of $\mathbb{R}^2$. A closed set contains it boundary points and a bounded set fits inside some finite disk in $\mathbb{R}^2$.

> **Thm/** If $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function on a closed bounded set D in $\mathbb{R}^2$ then f attains an absolute maximum value and an absolute minimum value somewhere in D.

Advice: to find extreme values for continuous f on D we

1.) find the critical points (where $\nabla f = 0$ or ∇f d.n.e.) and evaluate

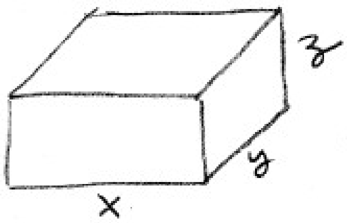
2.) find extreme values of f on the boundary of D (which I call ∂D)

3.) The values from 1.) and 2.) compare and choose biggest/smallest.

(see hwk for example)

$\boxed{E83}$ Consider a company that accepts only rectangular boxes whose length and girth (the perimeter of a cross section) do not sum over 108". Find the dimensions of an acceptable box of largest volume.

$$V = xyz$$



$$\underbrace{x}_{\text{length}} + \underbrace{2y + 2z}_{\text{girth}} = 108$$

$$x = 108 - 2y - 2z$$

$$V(y,z) = (108 - 2y - 2z)yz = 108yz - 2y^2z - 2yz^2$$

$$V_y = 108z - 4yz - 2z^2 = 0$$
$$V_z = 108y - 2y^2 - 4yz = 0$$

looking for critical points.

$$z(108 - 4y - 2z) = 0$$
$$y(108 - 2y - 4z) = 0$$

Now we need these to be simultaneously zero. Notice $z=0$ and $y=0$ gives one solution. Or we could have $108 - 4y - 2z = 0$ AND $108 - 2y - 4z = 0$. Or we could have $z=0$ and $108-2y-4z=0$ or $y=0$ and $108-4y-2z=0$.

$\boxed{\text{E83 continued}}$ We found $(0,0)$ our first critical point. There are three more,

(i) $108 - 4y - 2z = 0$ $\Rightarrow$ $\begin{pmatrix} 216 - 8y - 4z = 0 \\ -\,(108 - 2y - 4z = 0) \end{pmatrix}$

$108 - 2y - 4z = 0$

$108 - 6y = 0 \;\therefore\; y = \frac{108}{6} = \frac{54}{3} = 18.$

$z = 54 - 2y = 54 - 36 = 18 = z$

$\therefore$ $(18,18)$ another critical pt.

(ii.) $z = 0$

$108 - 2y - 4z = 0 \Rightarrow 108 - 2y = 0 \Rightarrow y = 54 \quad \therefore$ $(54, 0)$ critical pt.

(iii) $y = 0$

$108 - 2z - 4y = 0 \Rightarrow 108 - 2z = 0 \Rightarrow z = 54 \quad \therefore$ $(0, 54)$ critical point

we have exposed that $(\nabla V)(y,z) = 0$ yields 4 sol^{n}'s, $(0,0)$, $(18,18)$ $(0,54)$ and $(54,0)$, these points must give the min/max. values. Calculate

$$V_{yy} = -4z \qquad V_{yz} = 108 - 4z$$

$$V_{zz} = -4y$$

$$D = 16yz - (108 - 4z)^2$$

Notice $V(0,0) = 0$, $V(0,54) = 0$ and $V(54,0) = 0$ thus we suspect that $(18,18)$ gives maximum volume, lets check

$$D(18,18) = 16(18)^2 - (108 - 72)^2 = 16(18)^2 - (36)^2 = 16(18)^2 - 4(18)^2 = 12(18)^2 > 0$$

and $V_{yy} = -4(18) < 0 \;\therefore\; V(18,18)$ is maximum.

Notice $x = 108 - 2y - 2z = 108 - 36 - 36 = 36$ thus

$$V = xyz = (36)(18)(18) = \boxed{11{,}664 \text{ in}^3 = \text{max volume subject to the constraint } x = 108 - 2y - 4z}$$

Remark: we just maximized a function of three variables x, y, z subject to a constraint. Our method was to substitute the constraint then treat it as a 2 variable min/max problem. There is a clever system for the problem in general, the method of Lagrange Multipliers, our next topic.

Lagrange Multipliers

Our goal is to find the extrema of $f(x,y,z)$ subject to a constraint condition $g(x,y,z)=0$. If f has an extreme value at $P=(x_0,y_0,z_0)$ then the curve $r(t)=\langle x(t), y(t), z(t)\rangle$ composed with f will have an extreme value at t_0, where $r(t_0)=(x_0,y_0,z_0)$ thus

$$0=\frac{d}{dt}\Big[f(x(t),y(t),z(t))\Big]\Big|_{t_0}=\left(\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial y}\frac{dz}{dt}\right)\Big|_{t_0}$$

$$=\underline{(\nabla f)(P)\cdot r'(t_0)=0}$$

On the other hand this curve can also be composed with g

$$\frac{d}{dt}\Big[g(x(t),y(t),z(t))\Big]=\frac{\partial g}{\partial x}\frac{dx}{dt}+\frac{\partial g}{\partial y}\frac{dy}{dt}+\frac{\partial g}{\partial z}\frac{dz}{dt}=(\nabla g)\cdot r'(t)=\frac{d}{dt}(k)=0,$$

where we get zero since $g(x,y,z)=k$, thus we also have $\underline{(\nabla g)(P)\cdot r'(t_0)=0}$. Therefore we deduce that $\nabla f=\lambda\nabla g$ at the extremum.

> **Method of Lagrange Multipliers:**
> Assuming that $f(x,y,z)$ has max/min values on the surface $g(x,y,z)=k$ with $\nabla g\neq 0$ we can find them as follows
>
> (i) set $\nabla f=\lambda\nabla g$ and use $g(x,y,z)=k$ to simplify and find sol^{n}'s.
>
> (ii.) evaluate the sol^{n}'s from (i) to see where f attains its min/max
>
> If $f=f(x,y)$ and $g=g(x,y)=k$ then do the same, just with 2-dim'l gradients.

$\boxed{E84}$ Let $f(x,y)=xy$ find extrema of f on the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$. I identify that $g(x,y)=x^2/8+y^2/2=1$. Consider then

$$\begin{aligned}\nabla f=\lambda\nabla g &\Rightarrow \langle y,x\rangle=\lambda\langle x/4, y\rangle\\ &\Rightarrow 4y=\lambda x \quad\text{and}\quad x=\lambda y\\ &\Rightarrow y=\frac{\lambda}{4}(\lambda y)\\ &\Rightarrow y(1-\lambda^2/4)=0\\ &\Rightarrow \underline{y=0 \quad\text{or}\quad \lambda=\pm 2.}\end{aligned}$$

$\searrow$

E84 continued We've gathered that $y=0$ or $\lambda = \pm 2$ yield the extrema,

$y=0$ then $x = \lambda y$ and so $x=0$ as well but $(0,0)$ not on ellipse.

$\lambda = 2$ $x = 2y$ and $4y = 2x$ a.k.a. $y = \frac{1}{2}x$

$$\frac{x^2}{8} + \frac{y^2}{2} = \frac{x^2}{8} + \frac{1}{2}\frac{x^2}{4} = \frac{1}{4}x^2 = 1 \therefore x^2 = 4$$

$\therefore x = \pm 2$ and $y = \frac{1}{2}(\pm x) = \pm 1$ so $(-2,-1)$ or $(2,1)$

$f(-2,-1) = (-2)(-1) = 2$ while $f(2,1) = 2(1) = 2$.

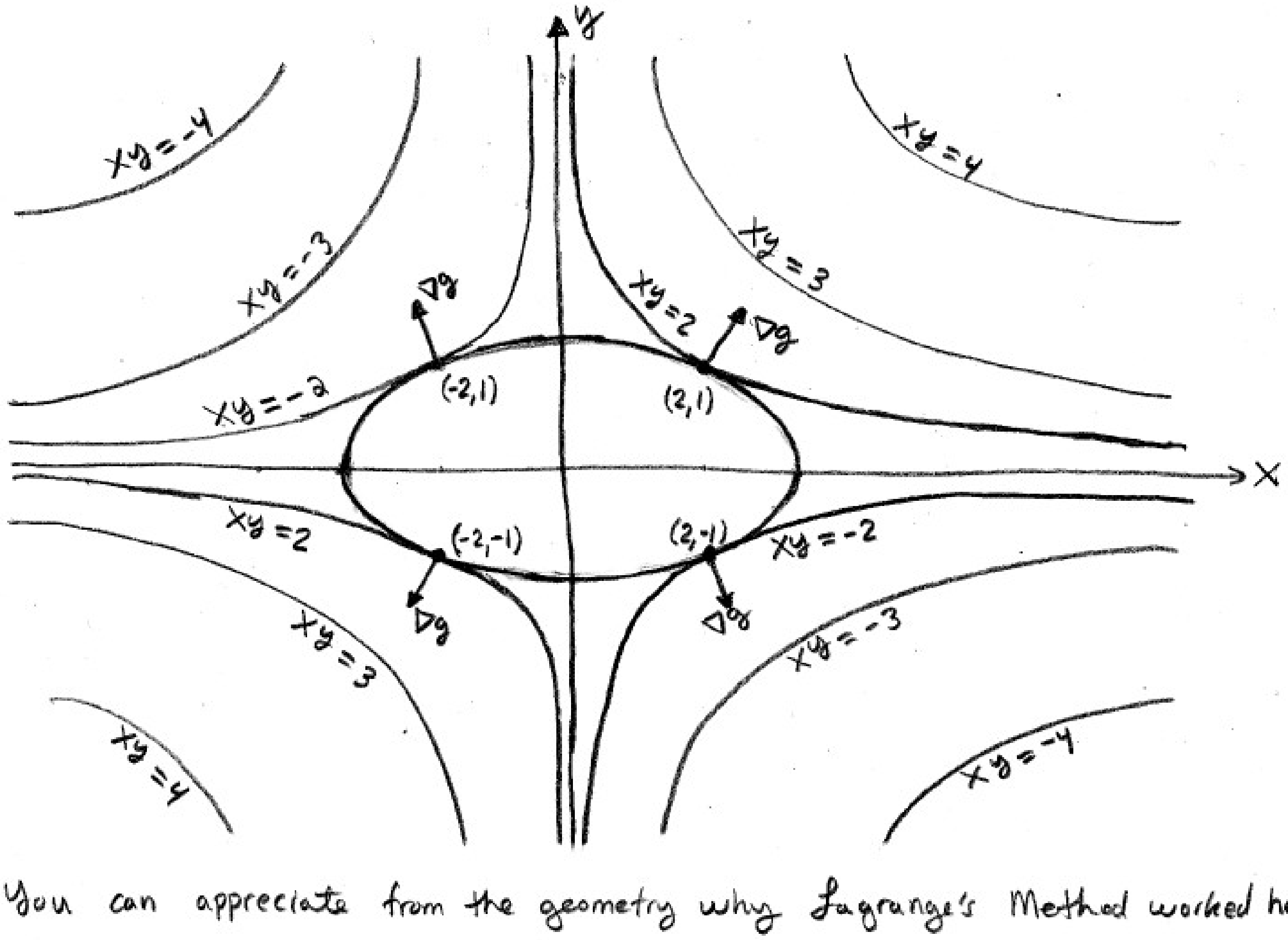
$\lambda = -2$ $x = -2y$ and $4y = -2x$ that is $y = -\frac{1}{2}x$

$$\frac{x^2}{8} + \frac{y^2}{2} = \frac{x^2}{8} + \frac{1}{2}\frac{x^2}{4} = \frac{1}{4}x^2 = 1 \therefore x^2 = 4$$

$\therefore x = \pm 2$ and $y = \frac{1}{2}(\mp x) = \mp 1$ so $(-2,1)$ or $(2,-1)$

$f(-2,1) = -2$ while $f(2,-1) = -2$.

The extreme values are 2 and -2. The max is 2 which is reached at $(-2,-1)$ and $(2,2)$ while the min. is obtained at $(-2,1)$ and $(2,-1)$.



You can appreciate from the geometry why Lagrange's Method worked here.

E85 Find the point on the plane $z = x + y$ that is closest to the point $(1,1,0)$. In other words minimize $f(x,y,z) = (x-1)^2 + (y-1)^2 + z^2$ subject to $g(x,y,z) = x + y - z = 0$.

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-1), 2(y-1), 2z \rangle = \lambda \langle 1, 1, -1 \rangle$$

$$\Rightarrow \begin{cases} 2(x-1) = \lambda \\ 2(y-1) = \lambda \\ 2z = -\lambda \end{cases}$$

$$\Rightarrow \lambda/2 = x - 1 = y - 1 = -z$$

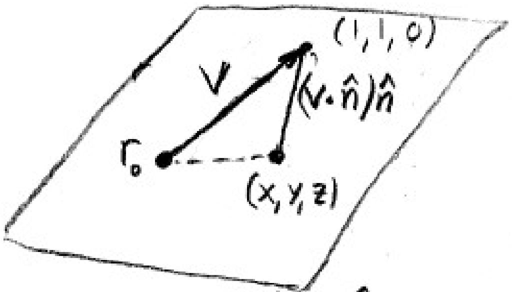
$$\Rightarrow x = y \quad \text{and} \quad z = 1 - y$$

$$\Rightarrow \text{since } z = x + y = 2y = 1 - y$$

$\therefore 3y = 1$ thus $y = 1/3 = x$ and $z = 1 - 1/3 = 2/3$.

$\boxed{\text{the closest point on the plane } z = x+y \text{ to the point } (1,1,0) \text{ is } (1/3, 1/3, 2/3)}$

<u>Lets check our answer geometrically:</u>



$$V = (1,1,0) - r_0$$

$$(V \cdot \hat{n})\hat{n} = \text{proj}_n(V)$$

$$(x,y,z) = (1,1,0) - (V \cdot \hat{n})\hat{n}.$$

We just need to find a normal of the plane and a point on the plane. Choose $n = \langle 1, 1, -1 \rangle$ so that $\hat{n} = \frac{1}{\sqrt{3}}\langle 1, 1, -1 \rangle$ and $r_0 = \langle 0, 0, 0 \rangle$. Hence,

$$V = (1,1,0)$$

$$\text{proj}_n(V) = \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle \cdot \langle 1,1,0 \rangle \hat{n} = \frac{2}{\sqrt{3}}\hat{n} = \frac{2}{3}\langle 1,1,-1 \rangle.$$

$$(1,1,0) - \text{proj}_n(V) = (1,1,0) - \tfrac{2}{3}(1,1,-1) = \boxed{(1/3, 1/3, 2/3)}$$

this was our intuitive sol$^{\underline{n}}$ we used in the early portion of this course, the closest point falls on the normal line connecting the point and the plane.

$\boxed{E86}$ A rectangular box without a lid is made from 12 square units of material. Find the maximum volume of such a box. That is maximize $V = xyz$ subject to $g = 2xz + 2yz + xy = 12$.

$$\nabla V = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2z + y, 2z + x, 2x + 2y \rangle$$

$$yz = \lambda(2z + y) \Rightarrow xyz = \lambda(2zx + yx)$$
$$xz = \lambda(2z + x) \Rightarrow xyz = \lambda(2zy + xy)$$
$$xy = \lambda(2x + 2y) \Rightarrow xyz = \lambda\underbrace{(2xz + 2yz) = \lambda(12 - xy)}_{\text{using } g = 0}$$

Thus $\lambda(2zx + xy) = \lambda(2zy + xy) = \lambda(12 - xy)$. We can divide by λ since $\lambda = 0 \Rightarrow xyz = 0$. Note then

$$2zx + xy = 2zy + xy = 12 - xy$$
$$\Rightarrow 2zx = 2zy \Rightarrow x = y \quad \text{(note } z = 0 \text{ is not a useful value.)}$$

Next notice $\cancel{2zy} + xy = \cancel{2xz} + 2yz \Rightarrow y^2 = 2yz \therefore y = 2z$.

Then $2xz + 2yz + xy = 4z^2 + 4z^2 + 4z^2 = 12z^2 = 12$.

Hence $z = \pm 1$. Our material only comes in positive lengths so $z = 1$ hence $x = y = 2$. $\boxed{\text{The box is } 2 \times 2 \times 1.}$

<u>Remark</u>: there are a couple examples in your text I haven't stolen.

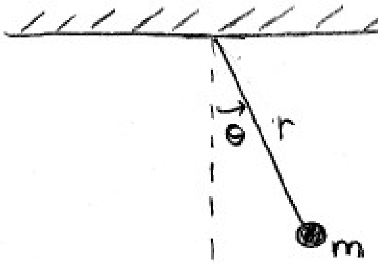
<u>Remark</u>: In the study of Lagrangian Mechanics if one has the constraint $g = 0$ then it can be implement by adding λg to the Lagrangian. $\mathcal{L} \longmapsto \mathcal{L} + \lambda g$. Then one finds the eq^{n}'s of motion and at the end puts $g = 0$. In that theory the Lagrange multipliers λ are found to be the forces needed to maintain the constraint eq^{n}'s on the motion of the physical body. It is one of the most beautiful chapters in Classical Mehanics, take PY 411 to see this in some detail, or buy an old copy of Goldstein's Classical Mechanics, its like \$10, its a classic. I sketch the method on the next pg (not req^{d} topic)

Digression: Lagrange Multipliers in Classical Mechanics:

You may ignore this if you wish. The Lagrangian in classical mechanics is the function $L = T - U$ where T = Kinetic energy and U = potential energy. The eq^{n}'s of motions follow from minimizing the action (Hamilton's Principle) and are called the Euler - Lagrange Eq^{n}'s

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad \longleftarrow \left(\begin{array}{c}\text{replaces Newton's Law} \\ \text{conceptually, energy} \\ \text{not force is primary}\end{array}\right)$$

To impose constraints one may add Lagrange Multipliers to encode those constraints. I'll illustrate with the simple pendulum, the constraint is $r = l$ a.k.a. $f = r - l = 0$.



$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$U = -mgr\cos\theta$$
$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$

Then the technique of Lagrange Multipliers adds the term on the RHS

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda\frac{\partial f}{\partial r} \quad \Rightarrow \quad m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \underbrace{\lambda}_{\text{Lagrange multiplier}}$$

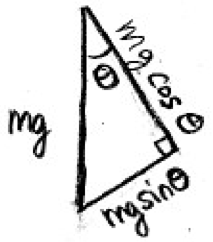
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) + mgr\sin\theta = 0$$

Then we have the following eq^{n}'s to solve, subject to $\underline{r = l}$

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \lambda$$
$$\frac{d}{dt}\left(mr^2\dot{\theta}^2\right) + mgr\sin\theta = 0$$

This example is kinda silly since the constraint is so trivial, once we apply $r = l$ we find $\lambda = -ml\dot{\theta}^2 - mg\cos\theta$ and the eq^{n}

$$\frac{d}{dt}\left(ml^2\dot{\theta}^2\right) + mgl\sin\theta = 0.$$



These can be solved for small θ where $\sin\theta \cong \theta$. It can be seen that λ is the force that enforces the constraint $r = l$. Generally Lagrange multipliers allow us to solve the eq^{n}'s of motion subject to some geometric (even time dependent) constraint w/o knowing the forces that cause the motion to be constrained. Neat thing is the method shows what the forces are.