Integral Vector Calculus: Overview

We have studied how to differentiate vector fields. We saw that three main operations were of interest. The gradient which created a vector field from a function, the curl which generated a new vector field as its output and the divergence which outputted a scalar function given a vector field input. Those vector derivatives will appear again as we continue.

It turns out there are two ways to integrate a vector field

(1.) $\int_C \vec{F} \cdot d\vec{r}$ = line integral of F along curve C.

(2.) $\int_S \vec{F} \cdot d\vec{S}$ = surface integral of F over surface S.

The line integral is used to calculate work done by $\vec{F}$ along C and the surface integral is used calculate the net "flux" of $\vec{F}$ that "cuts" through the surface S. Of course there are many other applications, a few of which we will study. Finally, we conclude our study of vector calculus with the powerful Green's, Stokes and Divergence Th^{m}'s.

Curves verses Paths

I may have inadvertently confused these two concepts before, but now we must be careful to distinguish.

Def^n/ A <u>curve</u> is a collection of points in $\mathbb{R}^n$, which can be parametrized by a path. If the curve has a direction we say it is <u>oriented</u>

Def^n/ We say $r : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$ is a <u>path</u>. It is a parametrization of the curve $r(I) \subset \mathbb{R}^n$. If C is oriented then a path $r(t)$ is said to be <u>consistently oriented</u> if it shares the same direction.

- A path or curve is closed if it forms a loop, so $r(a) = r(b)$.

E137 The curve $C = \{(x,y) \mid x^2+y^2=1\}$ can be oriented in the counter-clockwise direction ↺. Its a circle. Notice

$$r_1(t) = (\cos t, \sin t) \quad : \text{consistently oriented}$$
$$r_2(u) = (\cos(-u), \sin(-u)) \quad : \text{oppositely oriented.}$$

We will say r_2 is consistently oriented with $-C$. That is "$-C$" means C with the opposite direction attached (↻).

- We need a few definitions to lead up to the def$^{\underline{n}}$ of $\int \vec{F} \cdot d\vec{r}$.

Def$^{\underline{n}}$/ $$\int_C f(x,y,z)\,ds \equiv \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

where we assume C is a curve with non intersecting parametrization $r(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$. Generally if f is a function of $(x_1, x_2, \dots, x_n)$ then if $r : [a,b] \to C \subset \mathbb{R}^n$,

$$\int_C f(x_1, x_2, \dots, x_n)\,ds \equiv \int_a^b f(r(t)) \left|\frac{dr}{dt}\right| dt$$

so in $n=2$ we just drop the z-terms.

this integral is quite similar to the arclength integral which was $s = \int_a^b |r'(t)|\,dt$. The integral of f along C is calculating the area of the curtain under C, see FIGURE 2 on pg. 913 of Stewart.

E138 Let $C = \{(x,y) \mid x^2+y^2=16,\ x \ge 0\}$. Calculate $\int_C xy^4\,ds$.
We need a parametrization of C, an easy geometrically motivated choice is $x = 4\cos\theta$, $y = 4\sin\theta$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned}
\int_C xy^4\,ds &= \int_{-\pi/2}^{\pi/2} (4\cos\theta)(4\sin\theta)^4 \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta \\
&= (4)^6 \int_{-\pi/2}^{\pi/2} \sin^4\theta \cos\theta\,d\theta \\
&= \frac{4^6}{5} \sin^5(\theta)\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{4^6}{5}\left(1^5 - (-1)^5\right) = \boxed{\frac{8192}{5}}
\end{aligned}$$

Integrals with respect to dx, dy and dz

Defn/

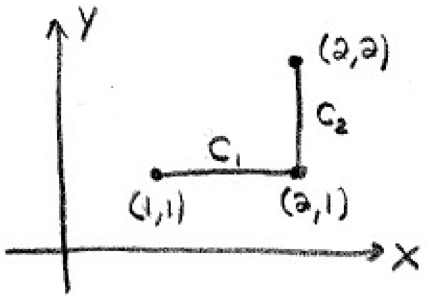
$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,\frac{dx}{dt}\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,\frac{dy}{dt}\,dt$$

$$\int_C f\,dx + \int_C g\,dy \equiv \int_C f\,dx + g\,dy$$

- Assuming C is parametrized by the path $r(t) = (x(t), y(t))$ which goes from $r(a)$ to $r(b)$.

E139 Let C be an L-shaped curve graphed below. Calculate the x-integral of $f(x,y) = xy^2$ along C. We have to break this up into pieces, I suppose our definition should have mentioned that we take the sum of the parts if we cannot globally parametrize the curve C as in here.



The curves C_1 & C_2 are easily parametrized,

$$C_1 : \quad r_1(t) = \langle 1,1\rangle + t\langle 1,0\rangle = \langle 1+t, 1\rangle, \quad 0 \le t \le 1$$

$$C_2 : \quad r_2(t) = \langle 2,1\rangle + t\langle 0,1\rangle = \langle 2, 1+t\rangle, \quad 0 \le t \le 1$$

Thus

$$\int_C xy^2\,dx = \int_{C_1} xy^2\,dx + \int_{C_2} xy^2\,dx$$

$$= \int_0^1 (1+t)\frac{d}{dt}(1+t)\,dt + \underbrace{\int_0^1 2(1+t)^2 \cancelto{0}{\frac{d}{dt}(2)}\,dt}_{\text{this is because } dx = 0 \text{ along } C_2.}$$

$$= \int_0^1 (1+t)\,dt$$

$$= \left(t + \tfrac{1}{2}t^2\right|_0^1$$

$$= 1 + \tfrac{1}{2} - 0$$

$$= \boxed{3/2}$$

Def$^{\underline{n}}$/ The integrals along x, y or z of $f(x,y,z)$ is,

$$\int_C P\,dx + Q\,dy + R\,dz \equiv \int_a^b P(r(t))\frac{dx}{dt}dt + \int_a^b Q(r(t))\frac{dy}{dt}dt + \int_a^b R(r(t))\frac{dz}{dt}dt$$

where C is parametrized by $r(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$.

E140 Let C be the helix $r(t) = \langle a\cos(t), a\sin t, bt\rangle$, $0 \leq t \leq 2\pi$. Lets calculate, dx, dy, dz to begin

$$x = a\cos t \qquad dx = -a\sin t\,dt$$
$$y = a\sin t \qquad dy = a\cos t\,dt$$
$$z = bt \qquad dz = b\,dt$$

to calculate integrals over dx, dy, dz you simply convert everything to t as indicated in the def$^{\underline{n}}$.

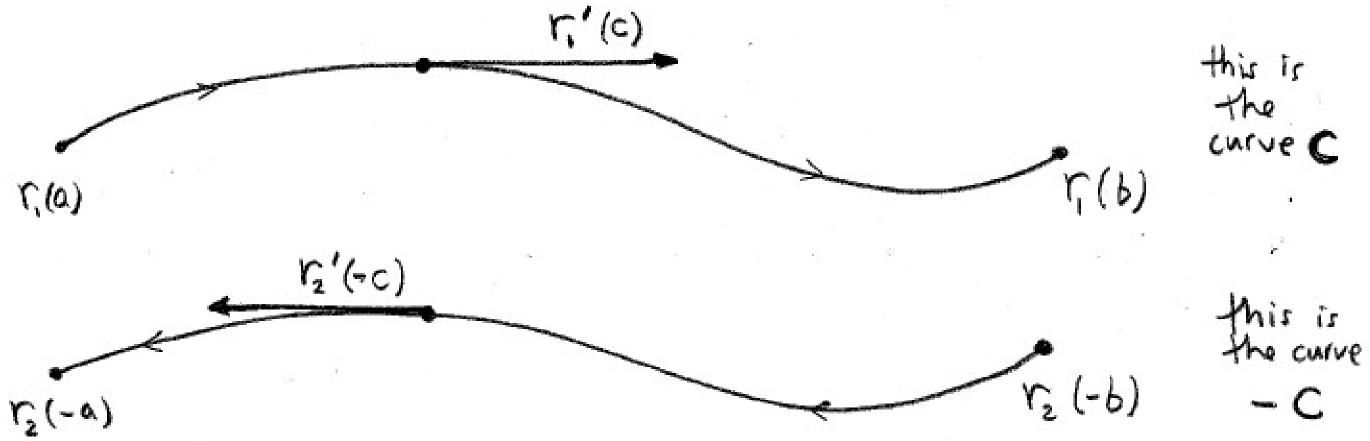
$$\int_C x\,dy - y\,dx + x\,dz = \int_0^{2\pi}(a\cos t)(a\cos t\,dt) - (a\sin t)(-a\sin t\,dt) + (a\cos t)(b\,dt)$$

$$= \int_0^{2\pi}(a^2 + ab\underbrace{\cos t}_{0})dt$$

$$= \boxed{2\pi a^2}$$

• note $\cos t$ will average to zero over a whole period

Remark: if $r_1(t)$ parametrizes the oriented curve C consistently we can see that $r_2(u) = r_1(-u)$ gives C the opposite orientation. In other words $-C$ has consistent oriented parametrization $r_2(u)$. If $a \leq t \leq b$ for $r_1(t)$ then $-b \leq u \leq -a$ for $r_2(u)$.



Integrals over C verses −C

We note that the x, y, z integrals along C differ from those along −C however the integral with respect to arclength is orientation independent.

$$\boxed{\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \quad \Bigg| \quad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy \quad \Bigg| \quad \int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds}$$

__Proof:__ follows from the def$^{\underline{n}}$ and our remark on 388 $\underset{[C]}{r_2(t)} = \underset{[-C]}{r_1(-t)}$.

$$\int_{-C} f(x,y)\,dx = \int_{-b}^{-a} f(r_2(u))\frac{dx}{du}\,du$$

$$u = -t \text{ and } du = -dt$$
$$\frac{dx}{du} = \frac{dt}{du}\frac{dx}{du} = -\frac{dx}{du}$$
$$u = -a = -t \Rightarrow \underline{t=a}$$
$$u = -b = -t \Rightarrow \underline{t=b}$$

$$= \int_b^a f(r_2(-t))\frac{dx}{dt}\,dt$$

$$= -\int_a^b f(r_1(t))\frac{dx}{dt}\,dt$$

$$= -\int_C f(x,y)\,dx.$$

Notice that $ds = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du = -\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ however the other minus from switching bounds compensates this sign.

$$\int_{-C} f(x,y)\,ds = \int_{-b}^{-a} f(r_2(u))\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du$$

$$= -\int_b^a f(r_2(-t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_a^b f(r_1(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_C f(x,y)\,ds.$$

- So we know how to integrate with respect to arclength, and dx, dy, dz. Lets move on to the most important definition which happens to be related to these (in my oppinion) weird integrals.

Def^{n}/ Let F be a continuous vector field defined on a smooth oriented curve C with parametrization $r(t)$, $a \leq t \leq b$. The line integral of F along C is

$$\int_C F \cdot dr \equiv \int_a^b F(r(t)) \cdot r'(t)\, dt = \int_C F \cdot T\, ds$$

Let me justify that the last equality holds,

$$\int_a^b F(r(t)) \cdot r'(t)dt = \int_a^b F(r(t)) \cdot \frac{r'(t)}{|r'(t)|} |r'(t)|\, dt \qquad \text{: multiplied by one, } r(t) \text{ nonstop.}$$

$$= \int_a^b F(r(t)) \cdot T(t)\, |r'(t)|\, dt \qquad : T(t) \equiv \frac{r'(t)}{|r'(t)|}$$

$$= \int_C F \cdot T\, ds.$$

Now you might ask why I insisted that C is oriented. Afterall we just argued that $\int_C f(x,y)ds = \int_{-C} f(x,y)ds$. The subtle point here is that $F \cdot T$ depends on the orientation of the curve: $F \cdot T\big|_C = -F \cdot T\big|_{-C}$, that is the tangent vector to C points in the <u>opposite direction</u> to the tangent vector to $-C$. $\therefore$ $\boxed{\int_C F \cdot dr = -\int_{-C} F \cdot dr}$ — (*)

<u>Remark:</u> the line integral matches up with $\int P dx + Q dy$ in 2-d and in 3-d $\int P dx + Q dy + R dz$, take $F = \langle P, Q, R \rangle$ then

$$\int_C F \cdot dr = \int_a^b \langle P, Q, R \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle dt$$

$$= \int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt$$

$$= \int_C P dx + Q dy + R dz.$$

Thus $\int_C F \cdot dr$ and $\int_C P dx + Q dy + R dz$ are just alternate notations for the same concept. I prefer $\int_C F \cdot dr$ because it is <u>coordinate</u> <u>independent</u>

E141 Let $F = \langle xy, y, -yz \rangle$ and suppose the oriented curve C has parametrization $r(t) = \langle t, t^2, t \rangle$, $0 \leq t \leq 1$. Calculate the line integral of F along C.

$$\int_C F \cdot dr = \int_0^1 F(t, t^2, t) \cdot r'(t)\, dt$$

$$= \int_0^1 \langle t^3, t^2, -t^3 \rangle \cdot \langle 1, 2t, 1 \rangle dt$$

$$= \int_0^1 (\cancel{t^3} + 2t^3 - \cancel{t^3})\, dt$$

$$= \frac{2}{4} t^4 \Big|_0^1 = \boxed{\frac{1}{2}}$$

E142 Suppose $F = \langle xy, y, -yz \rangle$ is a __force field__ and some object travels the path $r(t) = \langle t, t^2, t \rangle$ during the time $0 \leq t \leq 1$. Find the __work__ done by F on the object. Think infinitesimally,

$$dW = F \cdot dr$$

this gives the little bit of work dW done by F as the object traverses dr. the dot-product is needed to pick the component of F along the direction of motion.

$$\therefore \boxed{W \equiv \int_C F \cdot dr} \quad (*)$$

therefore we have calculated the work already in E141 it is $\boxed{W = 1/2}$.

__Remark__: this is a generalization of $\int F(x)\,dx$ which gave us the work done by F along x, and $W = F \cdot d$ which worked for __constant forces__. The definition $(*)$ works for arbitrary forces over any old 3-d curve through space. I consider it to be the real definition of work.

More Examples:

$\boxed{E143}$ In $\boxed{E142}$ we considered $F = \langle xy, y, -yz \rangle$ and our path went from $(0,0,0)$ to $(1,1,1)$. Suppose we let the object travel a different path between these points, say $r(t) = \langle t, t^3, t^5 \rangle$, $0 \leq t \leq 1$. (C_2) What is the <u>work</u> done by F in this case?

$$W = \int_{C_2} F \cdot dr$$

$$= \int_0^1 \langle t^4, t^3, -t^8 \rangle \cdot \langle 1, 3t^2, 5t^4 \rangle \, dt$$

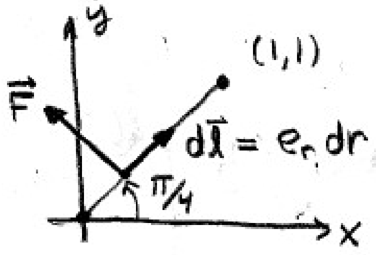
$$= \int_0^1 (t^4 + 3t^5 - 5t^{12}) \, dt$$

$$= \left[\frac{1}{5}t^5 + \frac{3}{6}t^6 - \frac{5}{13}t^{13} \right|_0^1$$

$$= \frac{1}{5} + \frac{1}{2} - \frac{5}{13} = \boxed{41/130}$$

<u>Remark:</u> we find that the work depends on the path taken, this is a path-dependent force. We'll see that a force must be <u>conservative</u> if the work done is to be path-independent. Notice that $\nabla \times F \neq 0$ for this force $\therefore \nexists f$ such that $F = \nabla f$. More on this later.

$\boxed{E144}$ Let $\vec{F} = \theta e_\theta$ where $r = \sqrt{x^2+y^2}$ and e_θ is unit-vector in the direction of increasing θ. Consider the radial path from $(0,0)$ to $(1,1)$, call it C.



$$\int_C \vec{F} \cdot d\vec{l} = \int_0^{\sqrt{2}} \left(\frac{\pi}{4} e_\theta\right) \cdot (e_r \, dr)$$

$$= 0 \quad \text{since } e_\theta \cdot e_r = 0$$

- this example illustrates the more geometrical method of evaluating path-integrals, you determine the infinitesimal line element $d\vec{l}$ then integrate as indicated.

$\boxed{E145}$ Consider $F = \langle 0, 0, -mg \rangle = -mg\,\hat{k}$. Find the work done as the mass m travels up the slope one helix a height h. The appropriate parametrization is

$$r(t) = \langle a\cos t,\ a\sin t,\ t \rangle \qquad 0 \le t \le h. \quad (C)$$

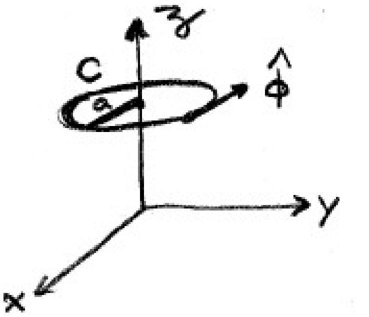
$$r'(t) = \langle -a\sin t,\ a\cos t,\ 1 \rangle = \frac{dr}{dt}$$

$$dr = (-a\sin t\,\hat{i} + a\cos t\,\hat{j} + \hat{k})\,dt$$

$$F \cdot dr = -mg\hat{k} \cdot (-a\sin t\,\hat{i} + a\cos t\,\hat{j} + \hat{k})\,dt = -mg\,dt$$

$$W = \int_C F \cdot dr = \int_0^h -mg\,dt = \boxed{-mgh = W}$$

$\boxed{E146}$ Use physics cylindricals. (see 383). Suppose $\vec{B} = \frac{\mu_o I}{2\pi s}\hat{\phi}$ where I, μ_o = constant whereas $s = \sqrt{x^2+y^2}$. Let C be a circle about the z-axis with radius a at $z = z_o$, oriented by $\hat{\phi}$.



$$d\vec{l} = \hat{\phi}\,a\,d\phi$$

$$\vec{B} \cdot d\vec{l} = \frac{\mu_o I}{2\pi s}\hat{\phi} \cdot \hat{\phi}\,a\,d\phi = \frac{a\mu_o I}{2\pi s}d\phi$$

$$\therefore \int_C \vec{B} \cdot d\vec{l} = \int_0^{2\pi} \frac{a\mu_o I}{2\pi a}d\phi = \frac{\mu_o I}{2\pi}\int_0^{2\pi} d\phi = \mu_o I.$$

$\boxed{E147}$ I'll behave and write the parametrization for C in $\boxed{E146}$. Its clear that the natural parameter is ϕ, $0 \le \phi \le 2\pi$,

$$r(\phi) = \langle a\cos\phi,\ a\sin\phi,\ z_o \rangle$$

$$\frac{dr}{d\phi} = \langle -a\sin\phi,\ a\cos\phi,\ 0 \rangle = a\hat{\phi}$$

$$\int_C \vec{B} \cdot dr = \int_0^{2\pi} \vec{B}(r(\phi)) \cdot r'(\phi)\,d\phi$$

$$= \int_0^{2\pi} \frac{\mu_o I}{2\pi a}\hat{\phi} \cdot a\hat{\phi}\,d\phi$$

$$= \frac{\mu_o I}{2\pi}\int_0^{2\pi} d\phi = \mu_o I$$

- the notation used in $\boxed{E146}$ demphasizes the parametrization. Conceptually they are the same though. The notation $d\vec{l}$ is $\approxeq$ to dr.

Remark: In E147 we found the relation

$$\int_C \vec{B}\cdot d\vec{l} = \mu_o I$$

the physical intepretation is that this is the magnetic field $\vec{B}$ which is generated by the current $\vec{I} = I\hat{k}$ along the z-axis. This is Ampere's Law which holds for steady currents. Generally in the magnetostatic case $I = \int_S J\cdot d\vec{A}$ where S is the surface with boundary C. When the current is not steady the situation is complicated by radiative effects.... You can take PY 414-415 to study E&M in depth.

Def^n/ When C is a closed path the quantity $\int_C F\cdot dr$ is called the circulation of F along C.

We found the circulation of $\vec{B}$ along C is proportional to the electric current. If $\vec{V}$ were the velocity field of a fluid then $\int_C \vec{V}\cdot dl$ gives the circulation of fluid along the closed curve C, basically it measures how the fluid flows in step with the curve. If the fluid flow was everywhere perpendicular to C then the circulation would be zero. Many early advances in Electricity and Magnetism (E&M) stemmed from careful study of fluid flow and its analogy to E&M. Faraday & Maxwell had much knowledge about fluid flow while they thought out the structure of E&M. Its unfortunate the public extended the analogy to worry about electricity literally "dripping" out of wires and such, I digress.

Overall Point: Line integrals can represent many physical concepts. Geometrically, $\int_C F\cdot dr$ measures how the vector field F lines up with C. The more in line the bigger, if $F \perp C$ everywhere then we get zero.

This theorem helps us understand the importance of the concept of the "conservative vector field". In E124, E125 we saw that certain vector fields could be written as the gradient of a scalar function. We defined earlier on 360 that F is conservative if $\exists f$ such that $F = \nabla f$. Then on 370 we found that if $F = \nabla f$ then $\nabla \times F = 0$. We say F is irrotational iff $\nabla \times F = 0$. Then on 371 we learned $\nabla \times F = 0 \not\Rightarrow F = \nabla f$ for some f, in all cases. However, if dom(F) is simply connected or all of $\mathbb{R}^3$ then $\nabla \times F = 0 \Leftrightarrow F = \nabla f$ for some f. Given a conservative vector field F we found how to calculate f such that $F = \nabla f$. The little "f" is called a potential function for F. Lets see what line integrals have to do with this story,

Th$^{\underline{m}}$/ Let C be an oriented, smooth (nonstop) curve given by $r(t)$ $a \le t \le b$. Let f be a differentiable function whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

"Fundamental Th$^{\underline{m}}$ for Line Integrals"

- We'll prove it after a few examples & comments.

Corollary: Given C as above and $F = \nabla f$ then

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(b)) - f(r(a)).$$

E148 in E145 on 393 we considered $F = \langle 0, 0, -mg \rangle$. We found that $\int_C F \cdot dr = -mgh$ for the helix C which had starting point $r(0) = \langle a, 0, 0 \rangle$ and ending point $r(h) = \langle a\cos h, a\sin h, h \rangle$. Notice that the potential function for this F is easy to find by inspection,

$$f = -mgz. \quad \text{observe} \quad \nabla f = \langle 0, 0, -mg \rangle \overset{\checkmark}{=} F$$

Lets check and see if the FTC for line integrals works,

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(h)) - f(r(0)) = -mgz \Big|_{r(0)}^{r(h)} = \boxed{-mgh}$$

$\boxed{E149}$ we found in $\boxed{E130}$ on 372 that $F = \langle 2x+y, z\cos(yz)+x, y\cos yz \rangle$ has the potential function $f = x^2 + xy + \sin(yz)$. You can check that $F = \nabla f$. Let C be <u>any</u> smooth (nonstop) curve from $(0,0,0)$ to $(1,1,\pi/2)$ find $\int_C F \cdot dr$

$$\begin{aligned}\int_C F \cdot dr = \int_C \nabla f \cdot dr &= f(1,1,\pi/2) - f(0,0,0) \\ &= 1 + 1 + \sin(\pi/2) - 0 \\ &= \boxed{3}\end{aligned}$$

- Notice the curve C was basically arbitrary. We could have taken another curve $\tilde{C}$ and obtained $\int_{\tilde{C}} F \cdot dr = 3$. This independence of the curve taken is defined as follows,

> Def^n/ If $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$ for all smooth oriented curves inside the domain of F then we say that $\int_C F \cdot dr$ is independent of path, or F is a path-independent.

Not all vector fields are path independent, for example we found that $\int_C F \cdot dr = \frac{1}{2}$ in $\boxed{E141}$ on 391 yet $\int_{C_2} F \cdot dr = 41/130$ in $\boxed{E143}$ on 392. We say such a F is <u>path dependent</u>.

- So we should prove the FTC for line integrals before we forget,

<u>Proof</u>: Suppose C has parametrization $r(t) = \langle x(t), y(t), z(t) \rangle$

$$\begin{aligned}\int_C \nabla f \cdot dr &= \int_a^b (\nabla f)(r(t)) \cdot r'(t)\, dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt}\Big[f(x(t), y(t), z(t)) \Big] dt && \text{multivariate chain rule.} \\ &= \int_a^b \frac{d}{dt}\Big[f(r(t)) \Big] dt && \text{: just chaning notation.} \\ &= f(r(b)) - f(r(a)). // && \text{: using the FTC of one variable calculus.}\end{aligned}$$

the proof is simple, as usual we simply borrow from calc. I to make the really nontrivial step.

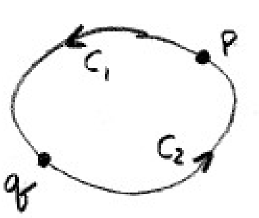
Independence of Path (§13.3)

We should generalize a bit and include piecewise smooth curves (which Stewart calls "paths", sorry there seems to be some disagreement as to what should be termed a path or curve.)

> **Thm/** $\int_C F\cdot dr$ is independent of path in $D \Leftrightarrow \int_C F\cdot dr = 0$ for all closed paths in D.

__Proof__: To prove $\Leftrightarrow$ we must prove $\Rightarrow$ and $\Leftarrow$.

$\Rightarrow$ Suppose $\int_C F\cdot dr$ is independent of path in D. Let C be a closed path in D. $C = C_1 \cup C_2$ where



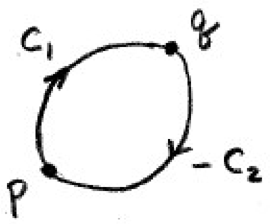
C_1 goes from P to q and
C_2 goes from q to P thus
$-C_2$ goes from P to q.

path independence $\Rightarrow \int_{C_1} F\cdot dr = \int_{-C_2} F\cdot dr = -\int_{C_2} F\cdot dr$ then,

$$0 = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr = \boxed{\int_C F\cdot dr = 0}$$

And C was an arbitrary closed curve so we have it for all closed curves in D.

$\Leftarrow$ Let C_1, C_2 be two curves in D from P to q then $-C_2$ goes from q to P. Thus $C_1 \cup \{-C_2\}$ is a closed curve in D. Thus by assumption



$$\int_{C_1\cup\{-C_2\}} F\cdot dr = 0$$

$$\|$$

$$\int_{C_1} F\cdot dr + \int_{-C_2} F\cdot dr = \int_{C_1} F\cdot dr - \int_{C_2} F\cdot dr = 0$$

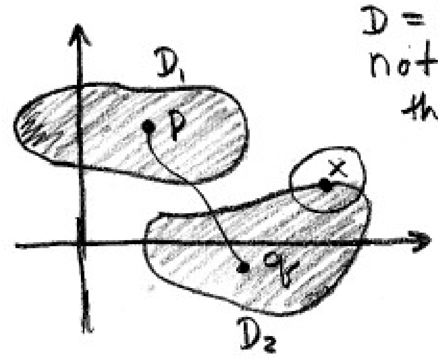
$$\therefore \boxed{\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr \quad \forall C_1, C_2 \text{ in } D.}$$

- the proof follows mainly from our result that reversing the orientation changes the sign, see 390 (*).

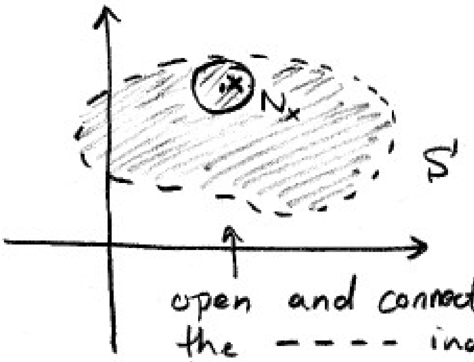
The proof of the Th^{m} on 397 and the one to follow here are worth considering because they are not terribly technical and they illustrate techniques that are conceptually important to path integrals.

Th^{m}/ Suppose F is continuous on an open connected region D. If $\int_C F \cdot dr$ is independent of path in D then F is a conservative vector field on D; that is $\exists$ a function f such that $F = \nabla f$

- In this course "connected" means path connected which means any two points in D can be connected by a path that is contained entirely in D. This is a topological concept.
- "Open" is also topological concept. D being open means that each point in D has a neighborhood about that point contained entirely in D.



$D = D_1 \cup D_2$ not connected, the path from p to q must go outside D. However, D_1 & D_2 are connected, but they're not open.



open and connected the - - - - indicates those points not in S. Each $x \in S$ can have a nbhd N_x about x entirely contained in S.

- Open, Closed, Bounded, Connected, Compact, Convergent, ... all of these ideas are treated seriously in Ma 425, 426 then ma 515 etc... We'll content ourselves with a few pictures.

Proof: Begin by picking $r_o = (a,b)$ a fixed point in D. We claim that the following is a potential function for F,

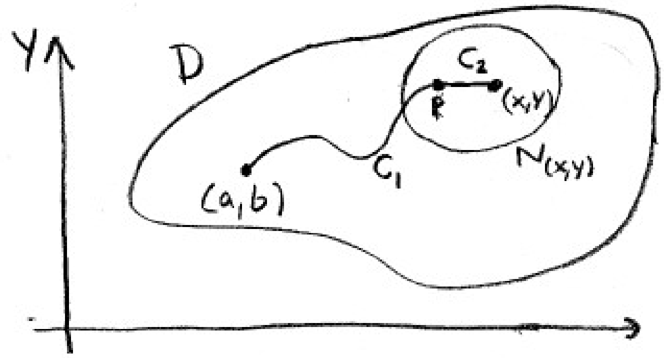
$$f(x,y) = \int_{(a,b)}^{(x,y)} F \cdot dr$$

Remark: In physics we define the electric potential V by

$$V(\vec{r}) \equiv -\int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{l} \implies E = -\nabla V$$

(the zero for the potential) (we'll prove this shortly upto that minus sign that differs)

<u>Proof Continued</u>: We claim $f(x,y) = \int_{(a,b)}^{(x,y)} F \cdot dr$. Lets draw a picture to start, notice because of path independence we didn't need to specify which path we take from (a,b) to (x,y). Consider a particular choice



$$C = C_1 \cup C_2$$

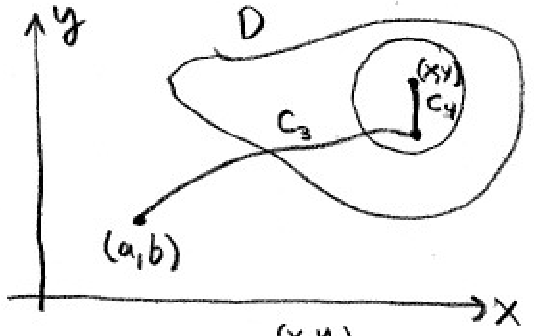
where C_1 goes from (a,b) to $P = (x_0, y) \in N_{(x,y)}$ then C_2 goes horizontally $(dy = 0)$ to (x,y).

$$f(x,y) = \underbrace{\int_{(a,b)}^{(x_0,y)} F \cdot dr}_{\text{independent of } x} + \underbrace{\int_{C_2} F \cdot dr}_{\text{has } dy=0}$$

Let $F = \langle P, Q \rangle$,

$$\frac{\partial}{\partial x}\left[f(x,y)\right] = \frac{\partial}{\partial x}\left[\int_{C_2} P\,dx + Q\,\underset{0}{dy}\right] = \frac{\partial}{\partial x}\left[\int_{x_0}^{x} P(t,y)\,dt\right] = P(x,y).$$

On the other hand we could choose a verticle path inside $N_{(x,y)}$.



This time $C = C_3 \cup C_4$ and C_3 goes from (a,b) to (x, y_0) then C_4 goes from (x, y_0) to (x,y) along C_4 we have $dx = 0$.

$$f(x,y) = \underbrace{\int_{(a,b)}^{(x,y_0)} F \cdot dr}_{\text{independent of } y} + \underbrace{\int_{C_4} F \cdot dr}_{\text{has } dx=0}$$

Let $F = \langle P, Q \rangle$

$$\frac{\partial}{\partial y}\left[f(x,y)\right] = \frac{\partial}{\partial y}\left[\int_{C_4} P\,\underset{0}{dx} + Q\,dy\right] = \frac{\partial}{\partial y}\left[\int_{y_0}^{y} Q(x,t)\,dt\right] = Q(x,y).$$

Therefore, we find that $\nabla f = \langle P, Q \rangle = F$ //.

<u>Remark</u>: this proof easily generalizes to $\mathbb{R}^n$, we use same argument of breaking up a path from $(a_1, a_2, \dots, a_n)$ to $(x_1, x_2, \dots, x_n)$ into a pieces so that C_1 doesn't depend on say x_k while C_2 has $dx_i = 0$ for $i \neq k$. It's not hard to picture for a sphere $N(x,y,z)$ in $\mathbb{R}^3$.

CONSERVATION OF ENERGY, WORK - ENERGY THEOREM

In §13.3 Th$^{\underline{m}}$(5) and Th$^{\underline{m}}$(6) describe conditions that gaurantee F is conservative. We discussed it in the 3-d case on 370 - 371 where we said F conservative $\Leftrightarrow$ $\nabla \times F = 0$ provided dom(F) is simply connected. We can shed a little more light on this discussion with the help of Stokes' Th$^{\underline{m}}$, but that must wait a bit. For now we consider the connection between conservative forces and the conservation of energy.

- Let $r(t)$ for $a \le t \le b$ describe the position of mass m as it travels along some path C. Further suppose that a conservative force F does work on m. We <u>define</u> the <u>potential energy function</u> to be U such that

$$F = -\nabla U \qquad \left(\begin{array}{l}\text{in contrast to } F = \nabla f \text{ where} \\ \text{we don't care about the sign}\end{array}\right)$$

Lets see why the <u>total energy</u> $E = K + U$ is conserved and also why physicists put in the minus sign for $F = -\nabla U$.

$$W \equiv \int_C F \cdot dr = -\int_C (\nabla U) \cdot dr = -\big[U(r(b)) - U(r(a))\big]$$

$$= \int_C ma \cdot dr \quad : \text{Newton's Law holds on } C.$$

$$= \int_a^b m \frac{d^2 r}{dt^2} \cdot \frac{dr}{dt}\, dt$$

$$\frac{d}{dt}[r' \cdot r'] = r'' \cdot r' + r' \cdot r''$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}[r' \cdot r'] = r'' \cdot r$$

$$\Rightarrow \frac{d}{dt}\left[\frac{m}{2}\frac{dr}{dt} \cdot \frac{dr}{dt}\right] = m \frac{d^2 r}{dt^2} \cdot \frac{dr}{dt}.$$

$$= \int_a^b \frac{d}{dt}\left[\frac{1}{2} m \frac{dr}{dt} \cdot \frac{dr}{dt}\right] dt$$

$$= \frac{1}{2} m |v(b)|^2 - \frac{1}{2} m |v(a)|^2 \quad : \text{identify this is the difference in Kinetic Energy from time b to time a.}$$

$$= K(b) - K(a)$$

We find, $K(b) - K(a) = -U(b) + U(a)$

$$\therefore \; U(a) + K(a) = U(b) + K(b)$$

$$\therefore \; \underline{E(a) = E(b)}$$

The total energy is conserved when the net force on some mass m is conservative.

Work Energy Theorem

Notice that our previous result had two lines of logic. One part held because $F = -\nabla U$, of course if F is not conservative we cannot hope the total energy for m is conserved. However, the other line of logic depended only on Newton's Law $F = ma$ (assuming m = constant). The Work Energy Th^{m} states that

$$\boxed{W = K(b) - K(a)}$$

this holds even for frictional forces etc....

$\boxed{E150}$ The Electric field $\vec{E}$ has $\vec{E} = -\nabla V$ in Electrostatics. The force on a charge q due to $\vec{E}$ is $\vec{F} = q\vec{E}$ then the potential energy $U = qV$ and

$$\vec{F} = -\nabla U = q(-\nabla V) = q\vec{E}$$

the electric potential V is the potential energy per unit charge. In the case $\vec{E} = (kQ/r^2)\hat{r}$ one has $V = -kQ/r$ if we assume that "$\mathcal{O} = \infty$". then

$$U(r) = \frac{-kqQ}{r}$$

And the conservation of energy for the charge q with mass m is

$$E_i = \frac{1}{2}mV_i^2 - \frac{kqQ}{r_i} = \frac{1}{2}mV_f^2 - \frac{kqQ}{r_f} = E_f$$

Other Electric Fields give different potential energy functions. But always we can calculate the potential from

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

and if you don't pick a zero for the potential (called $\mathcal{O}$) you can still calculate differences in the potential.