ADDITIONAL PROBLEMS ON CONSTRAINED PARTIAL DIFFERENTIATION: FROM COLLEY 1st Ed. pg. 157

These problems are req'd but not collected.

Let $w = f(x, y, z)$ be a differentiable function of x, y, and z. For example, suppose $w = x + 2y + z$. Regarding the variables x, y, and z as independent, we have $\partial w/\partial x = 1$ and $\partial w/\partial y = 2$. But now suppose that $z = xy$. Then x, y, and z are not all independent and, by substitution, we have that $w = x + 2y + xy$ so that $\partial w/\partial x = 1 + y$ and $\partial w/\partial y = 2 + x$. To overcome the apparent ambiguity in the notation for partial derivatives, it is customary to indicate the complete set of independent variables by writing additional subscripts beside the partial derivative. Thus

$$\left(\frac{\partial w}{\partial x}\right)_{y,z}$$

would signify the partial derivative of w with respect to x while holding both y and z constant. Hence x, y, and z are the complete set of independent variables in this case. On the other hand, we would use $(\partial w/\partial x)_y$ to indicate that x and y alone are the independent variables. In the case that $w = x + 2y + z$, this notation gives

$$\left(\frac{\partial w}{\partial x}\right)_{y,z} = 1, \qquad \left(\frac{\partial w}{\partial y}\right)_{x,z} = 2, \qquad \left(\frac{\partial w}{\partial z}\right)_{x,y} = 1.$$

If $z = xy$, then we also have

$$\left(\frac{\partial w}{\partial x}\right)_{y} = 1 + y, \qquad \left(\frac{\partial w}{\partial y}\right)_{x} = 2 + x.$$

In this way, the ambiguity of notation can be avoided. Use this notation in Exercises 21–25.

21. Let $w = x + 7y - 10z$ and $z = x^2 + y^2$.

(a) Find $\left(\frac{\partial w}{\partial x}\right)_{y,z}$, $\left(\frac{\partial w}{\partial y}\right)_{x,z}$, $\left(\frac{\partial w}{\partial z}\right)_{x,y}$, $\left(\frac{\partial w}{\partial x}\right)_{y}$, $\left(\frac{\partial w}{\partial y}\right)_{x}$.

(b) Relate $(\partial w/\partial x)_{y,z}$ and $(\partial w/\partial x)_y$ using the chain rule.

22. Repeat Exercise 21 where $w = x^3 + y^3 + z^3$ and $z = 2x - 3y$.

23. Suppose $s = x^2 y + xzw - z^2$ and $xyw - y^3 z + xz = 0$. Find

$$\left(\frac{\partial s}{\partial z}\right)_{x,y,w} \quad \text{and} \quad \left(\frac{\partial s}{\partial z}\right)_{x,w}.$$

24. Let $U = F(P, V, T)$ denote the internal energy of a gas. Suppose the gas obeys the ideal gas law $PV = kT$ where k is a constant.

(a) Find $\left(\frac{\partial U}{\partial T}\right)_{P}$. (b) Find $\left(\frac{\partial U}{\partial T}\right)_{V}$. (c) Find $\left(\frac{\partial U}{\partial P}\right)_{V}$.

25. Show that if x, y, z are related implicitly by an equation of the form $F(x, y, z) = 0$, then

$$\left(\frac{\partial x}{\partial y}\right)_{z} \left(\frac{\partial y}{\partial z}\right)_{x} \left(\frac{\partial z}{\partial x}\right)_{y} = -1.$$

This relation is used in thermodynamics. (Hint: Use Exercise 18.)