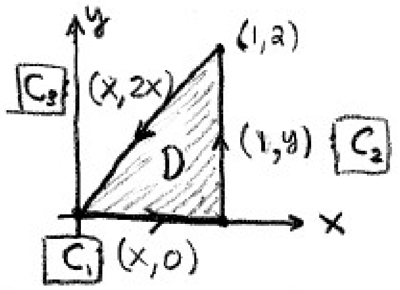
HIII

$\boxed{\S 13.4 \# 3}$ Let C be the $\triangle$ with vertices $(0,0)$, $(1,0)$ and $(1,2)$. Calculate the integral $I = \oint_C xy\,dx + x^2y^3\,dy$ directly & then by Green's Th^{m}.



$$I = \int_{C_1} x\underset{0}{y}\,dx + x^2\underset{0}{y^3}\,\underset{0}{dy} + \int_{C_2} xy\,\underset{0}{dx} + x^2y^3\,dy + \int_{C_3} xy\,dx + x^2y^3\,dy$$

$$= \int_0^2 y^3\,dy + \int_1^0 2x^2\,dx + \int_1^0 x^2(2x)^3\,2\,dx$$

$$= \frac{1}{4}y^4\Big|_0^2 + \frac{2}{3}x^3\Big|_1^0 + \frac{16}{6}x^6\Big|_1^0$$

$$= \frac{1}{4}(16) - \frac{2}{3} - \frac{8}{3} = \frac{48-40}{12} = \frac{8}{12} = \boxed{\frac{2}{3}}$$

(I've indicated that C_1, C_2, C_3 are parametrized by x, y, x respectively)

Now use Green's Th^{m}: $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ as an alternate sol^{n},

$$\oint_C xy\,dx + x^2y^3\,dy = \iint_D (2xy^3 - x)\,dA$$

$$= \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$$

$$= \int_0^1 \left(\frac{1}{2}x(2x)^4 - 2x^2\right)dx = \int_0^1 (8x^5 - 2x^2)\,dx = \frac{8}{6} - \frac{2}{3} = \boxed{\frac{2}{3}}$$

$\boxed{\S 13.4 \# 11}$ $C: x^2+y^2=4$. Let $D = \{(x,y) \mid x^2+y^2 \le 4\}$ so $\partial D = C$,

$$\oint_C y^3\,dx - x^3\,dy = \iint_D \left(\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right) dA$$

$$= \iint_D 3(-x^2 - y^2)\,dA$$

: use polar coordinates to integrate! $(x^2+y^2 = r^2)$

$$= \int_0^{2\pi}\int_0^2 -3r^3\,dr\,d\theta = (2\pi)(-3/4)(16) = \boxed{-24\pi}$$

$\boxed{\S 13.4 \# 12}$ Let C be the ellipse $x^2 + xy + y^2 = 1$. Let D be the filled in ellipse.

$$\oint_C \sin(y)\,dx + x\cos(y)\,dy \overset{(*)}{=} \iint_D (\cos(y) - \cos(y))\,dA = \iint_D 0\,dA = \boxed{0.}$$

((*) used Green's Th^{m})

Remark: Well isn't that convenient. And I just spent several minutes trying to picture that ellipse. Can you?

Hira

§13.4 #15 Let $\vec{F}(x,y) = \langle e^x + x^2y,\ e^y - xy^2 \rangle$. Let C be $x^2+y^2=25$ oriented clockwise. Let D be $x^2+y^2 \le 25$. Note that C is oriented opposite what is needed for Green's Thm.

$$\oint_C \vec{F}\cdot d\vec{r} = -\oint_{-C} \vec{F}\cdot d\vec{r}$$

: $-C$ is same curve oriented CCW

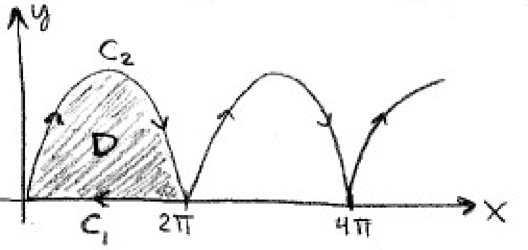
$$= -\iint_D \left(\frac{\partial}{\partial x}(e^y - xy^2) - \frac{\partial}{\partial y}(e^x + x^2y)\right) dA$$

$$= -\iint_D (-y^2 - x^2)\,dA$$

: use polar coordinates, $dA = r\,dr\,d\theta$.

$$= \int_0^{2\pi}\int_0^5 r^3\,dr\,d\theta = 2\pi\left(\frac{1}{4}(5)^4\right) = \frac{625\pi}{2}$$

§13.4#19 Consider the cycloid $x = t - \sin t$, $y = 1 - \cos t$. Find the area under one arch of the cycloid. Lets graph the cycloid to begin (I used my TI-89)



$C_1: x = 2\pi - t,\ y = 0,\ 0 \le t \le 2\pi$

$C_2: x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$

$C = C_1 \cup C_2$

Use the formula on pg. 936 to find the area,

$$A = \oint_{-C} x\,dy = -\oint_C x\,dy = -\int_{C_1} x\,\underbrace{dy}_{0} - \int_{C_2} x\,dy = -\int_0^{2\pi}(t - \sin t)\sin t\,dt$$

$$= -\int_0^{2\pi} \underbrace{t}_{u}\,\underbrace{\sin t\,dt}_{dv} + \int_0^{2\pi} \sin^2 t\,dt$$

$$= t\cos t\Big|_0^{2\pi} - \int_0^{2\pi} \cos(t)\,dt + \int_0^{2\pi} \frac{1}{2}(1 - \cos(2t))\,dt$$

$$= 2\pi - 0 + \frac{1}{2}t\Big|_0^{2\pi} - \frac{1}{4}\sin(2t)\Big|_0^{2\pi}$$

$$= \boxed{3\pi}$$

Remark: this problem illustrates the other subtle uses of Green's Thm. The generalization of this trick to Stoke's Thm. allows for the calculation of ugly surface areas by a relatively simple contour integral.

$\boxed{\S 13.7\#3}$ Consider $F = \langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\rangle$ and the surface $S: x^2+y^2+z^2 = 4$ with $z \geq 0$ oriented upwards, it has boundary $\partial S: x^2+y^2=4,\ z=0$ oriented CCW in the (xy)-plane.

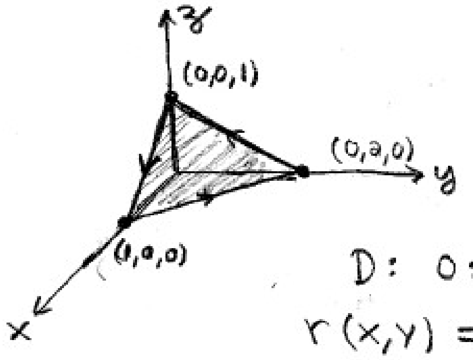
Notice ∂S has parametrization by θ naturally

$$x = 2\cos\theta \qquad dx = -2\sin\theta\, d\theta$$
$$y = 2\sin\theta \qquad dy = 2\cos\theta\, d\theta$$
$$z = 0 \qquad dz = 0$$

Now let Stokes' Th$^{\underline{m}}$ work it's magic,

$$\iint_S (\nabla\times F)\cdot d\vec{S} = \oint_{\partial S} F\cdot d\vec{r} = \oint_{\partial S} x^2e^{yz}dx + y^2e^{xz}dy + z^2e^{xy}dz$$
$$= \int_0^{2\pi}\left(x^2(-2\sin\theta) + y^2(2\cos\theta)\right)d\theta \quad : \text{note } z=0 \text{ on } \partial S \text{ this kills alot.}$$
$$= \int_0^{2\pi} 8(\sin^2\theta\cos\theta - \cos^2\theta\sin\theta)\,d\theta$$
$$= 8\left(\tfrac{1}{3}\sin^3\theta + \tfrac{1}{3}\cos^3\theta\right|_0^{2\pi}$$
$$= 0.$$

$\boxed{\S 13.7\#8}$ $F = \langle e^{-x}, e^x, e^z\rangle$, calculate $\int_C F\cdot dr$ where C is the boundary of $2x+y+2z=2$ with $x,y,z \geq 0$.



Study $2x+y+2z=2$ in the coord. planes,

$x=0: \ y = 2-2z$

$y=0: \ z = 1-x$

$z=0: \ y = 2-2x$

$D: \ 0 \leq x \leq 1, \ 0 \leq y \leq 2-2x$

$r(x,y) = \langle x, y, 1-x-\tfrac{1}{2}y\rangle$, $r: D \to S$ (D: param. space; S: the slanted triangle.)

$r_x = \langle 1, 0, -1\rangle$

$r_y = \langle 0, 1, -1/2\rangle$

$r_x \times r_y = \langle 1, 1/2, 1\rangle \ \therefore \ |r_x\times r_y| = \sqrt{1+\tfrac{1}{4}+1} = \sqrt{\tfrac{9}{4}} = \tfrac{3}{2}$.

Ok now I have the needed items to do a surface integral over S. But we still need to find the $\nabla\times F$,

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ e^{-x} & e^x & e^z \end{vmatrix} = \langle 0, 0, e^x\rangle$$

§13.7#8 Now finish it.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times F)\cdot d\vec{S}$$

$$= \iint_D \langle 0, 0, e^x\rangle \cdot \langle 1, 1/2, 1\rangle \, dA$$

$$= \int_0^1 \int_0^{2-2x} e^x \, dy\, dx$$

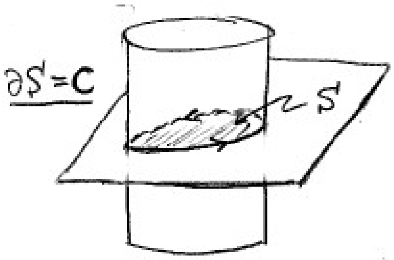
$$= \int_0^1 (2e^x - 2xe^x)\, dx$$

$$\int \underbrace{x}_{u}\underbrace{e^x dx}_{dV} = xe^x - \int e^x dx = xe^x - e^x + C$$

$$= 2(e^x - xe^x + e^x)\Big|_0^1$$

$$= 2(2e - e) - 2(2e^0) = \boxed{2e - 4}$$

§13.7#10 Find $\int_C F\cdot dr$ for $F = \langle xy, 2z, 3y\rangle$ and C is the intersection of the plane $x + z = 5$ and $x^2 + y^2 = 9$. This surface is easy to describe in cylindrical coordinates



$$z = 5 - x = 5 - r\cos\theta$$
$$0 \le \theta \le 2\pi$$
$$0 \le r \le 3$$

$$X(\theta, r) = \langle r\cos\theta, r\sin\theta, 5 - r\cos\theta\rangle$$

$$X_\theta = \langle -r\sin\theta, r\cos\theta, r\sin\theta\rangle$$

$$X_r = \langle \cos\theta, \sin\theta, -\cos\theta\rangle$$

$X_\theta \times X_r = \langle -r, 0, -r\rangle$ oups. we need $X_r \times X_\theta = r\langle 1, 0, 1\rangle$.

Now the curl,

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy & 2z & 3y \end{vmatrix} = \langle 1, 0, x\rangle$$

Thus, using Stoke's Th^{m} we find,

$$\int_C F\cdot dr = \iint_S \langle 1,0,x\rangle\cdot d\vec{S} = \int_0^{2\pi}\int_0^3 \langle 1,0,x\rangle\cdot\langle r,0,r\rangle\, dr\, d\theta \quad , \; \underline{x = r\cos\theta}$$

$$= \int_0^{2\pi}\int_0^3 (r + r^2 \underbrace{\cancel{\cos\theta}}_{0})\, dr\, d\theta = 2\pi\int_0^3 r\, dr = \boxed{9\pi}$$

full cycle ⇒ 0

§13.7#20 Use the identities we proved a few sections back namely $\nabla\times(F+G) = \nabla\times F + \nabla\times G$ & $\nabla\times(fF) = f(\nabla\times F) + (\nabla f)\times F$ plus Stoke's Th$^{\underline{m}}$ to derive the following,

H115

a.) $$\int_C (f\nabla g)\cdot dr = \iint_S (\nabla\times(f\nabla g))\cdot d\vec{S}$$
$$= \iint_S [f(\nabla\times\nabla g) + (\nabla f)\times(\nabla g)]\cdot d\vec{S}$$, note $\underline{\nabla\times\nabla g = 0}$.
$$= \iint_S [(\nabla f)\times(\nabla g)]\cdot d\vec{S}$$

b.) $$\int_C (f\nabla f)\cdot dr = \iint_S (\nabla f\times\nabla f)\cdot d\vec{S}$$ by part a.) with $f=g$.
$$= 0$$ since $\nabla f \parallel \nabla f$.

c.) $$\int_C (f\nabla g + g\nabla f)\cdot dr = \iint_S \nabla\times[f\nabla g + g\nabla f]\cdot d\vec{S}$$
$$= \iint_S (\nabla\times[f\nabla g] + \nabla\times[g\nabla f])\cdot d\vec{S}$$
$$= \iint_S (\cancel{\nabla f\times\nabla g} + \cancel{\nabla g\times\nabla f})\cdot d\vec{S}$$ (cross-product has $A\times B = -B\times A$)

§13.8#7 Calculate $\iint_S F\cdot dS$ via the divergence Th$^{\underline{m}}$, for $F = \langle 3xy^2, xe^z, z^3\rangle$ where $S = \partial B$ and B is the solid bounded by $x=-1$ and $x=2$ and $y^2+z^2=1$. Use modified cylindricals, $r \equiv \sqrt{y^2+z^2}$

$$x = x \ , \quad y = r\cos\varphi \ , \quad z = r\sin\varphi$$

where $-1\le x\le 2$ and $0\le r\le 1$ and $0\le\varphi\le 1$ gives B. Notice

$$\nabla\cdot F = 3y^2 + 0 + 3z^2 = 3r^2$$

Finally,

$$\iint_S F\cdot d\vec{S} = \iiint_B (\nabla\cdot F)\,dV$$
$$= \int_0^{2\pi}\int_0^1\int_{-1}^{2} 3r^2\, r\, dx\, dr\, d\varphi$$
$$= (3)(3/4)(2\pi)$$
$$= \boxed{9\pi/2}$$

I suppose technically we should calculate the appropriate Jacobean to prove this. Although, given the standard cylindrical results its just changing notation.

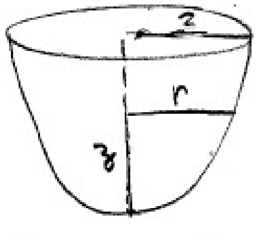
H116

$\boxed{\S 13.8\#8}$ $F = \langle x^3y, -x^2y^2, -x^2yz\rangle$ and let S be any old ugly surface satisfying conditions of the divergence Th^{m}.

$$\nabla\cdot F = 3x^2y - 2yx^2 - x^2y = 0 \Rightarrow \iint_S F\cdot ds = \iiint_B (\nabla\cdot F)\,dV = 0.$$

$\boxed{\S 13.8\#11}$ $F = \langle \cos z + xy^2,\ xe^{-z},\ \sin y + x^2z\rangle$

let S be the surface bounding B : $z = x^2+y^2$ & $z = 4$



$B \xrightarrow{} \quad 0 \le \theta \le 2\pi$

$0 \le r \le 2$

$r^2 \le z \le 4$

$$\nabla\cdot F = y^2 + 0 + x^2 = r^2$$

$$\iint_S F\cdot d\vec{S} = \iiint_B (\nabla\cdot F)\,dV \quad : \quad dV = r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^3\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (4r^3 - r^5)\,dr\,d\theta$$

$$= (2\pi)\left(16 - \tfrac{1}{6}(64)\right) = 2\pi\left(\frac{96-64}{6}\right) = \frac{64\pi}{6} = \boxed{\frac{32\pi}{3}}$$