§13.2#2 Calculate the line integral $\int_C (y/x)\,ds$ where C is a curve in the xy-plane parametrized by $x = t^4,\ y = t^3,\ 1/2 \le t \le 1$. By definition,

$$\begin{aligned}
\int_C (y/x)\,ds &\equiv \int_{1/2}^{1} \left(t^3/t^4\right)\sqrt{(4t^3)^2 + (3t^2)^2}\,dt \\
&= \int_{1/2}^{1} \frac{1}{t}\sqrt{16t^6 + 9t^4}\,dt \\
&= \int_{1/2}^{1} t\sqrt{16t^2+9}\,dt \\
&= \frac{2}{3}\,\frac{1}{32}\left(16t^2+9\right)^{3/2}\Big|_{1/2}^{1} \\
&= \frac{1}{48}\left[(25)^{3/2} - (13)^{3/2}\right] \\
&= \boxed{\frac{1}{48}\left[125 - 13\sqrt{13}\right]}
\end{aligned}$$

$\sqrt{t^4} = t^2$, factor it out.

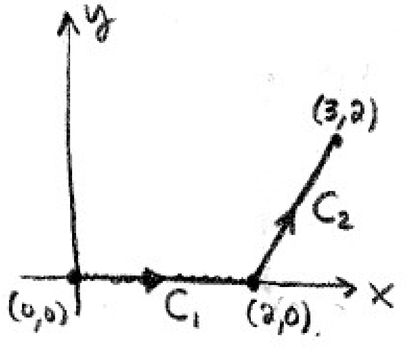
§13.2#4 Let C be the arc of $x = e^y$ from $(1,0)$ to $(e,1)$. Calculate

$$\begin{aligned}
\int_C x e^y\,dx &\equiv \int_0^1 e^y e^y \frac{d}{dy}\left(e^y\right)dy \\
&= \int_0^1 e^{3y}\,dy \\
&= \frac{1}{3}e^{3y}\Big|_0^1 = \boxed{\frac{1}{3}\left(e^3 - 1\right)}
\end{aligned}$$

(parametrize by y: $r(y) = \langle e^y, y\rangle$ for $0 \le y \le 1$.)

- here y played the role of the parameter which we abstractly call "t".

§13.2#5 Let C be the curve formed by two line segments.



Denote $C = C_1 \cup C_2$ where C_1 and C_2 are the line segments pictured below.

$C_1:\ r_1(t) = (t, 0) \qquad 0 \le t \le 2$

$C_2:\ r_2(\lambda) = (2,0) + \lambda(3-2, 2-0)$

$r_2(\lambda) = (2+\lambda, 2\lambda) \qquad 0 \le \lambda \le 1$

there are as always other parametrizations possible.

§13.2#5 We found a parametrization of $C = C_1 \cup C_2$ now calculate,

$$\int_C xy\,dx + (x-y)\,dy = \int_{C_1} x\underset{0}{y}\,dx + (x-\underset{0}{y})\,dy + \int_{C_2} xy\,dx + (x-y)\,dy$$

$$= \int_0^2 t\, y'(t)\,dt + \int_0^1 (2+\lambda)(2\lambda)\underbrace{x'(\lambda)}_{1}d\lambda + [(2+\lambda)-2\lambda]\underbrace{y'(\lambda)}_{2}d\lambda$$

$$= \int_0^2 t(0)\,dt + \int_0^1 (4\lambda + 2\lambda^2)\,d\lambda + (4-2\lambda)\,d\lambda$$

$$= \int_0^1 (2\lambda^2 + 2\lambda + 4)\,d\lambda$$

$$= 2\left(\tfrac{1}{3}\lambda^3 + \tfrac{1}{2}\lambda^2 + 2\lambda\right)\Big|_0^1$$

$$= 2\left(\tfrac{1}{3} + \tfrac{1}{2} + 2\right) = 2\left(\tfrac{2+3+12}{6}\right) = \boxed{\tfrac{17}{3}}$$

§13.2#8 Let C be line segment from $(0,6,-1)$ to $(4,1,5)$. We parametrize C as usual $r(t) = (0,6,-1) + t(4-0, 1-6, 5+1)$

$$x(t) = 4t \qquad dx/dt = 4$$
$$y(t) = -5t+6 \qquad dy/dt = -5$$
$$z(t) = 6t-1 \qquad dz/dt = 6$$

Notice $x^2 z = 16t^2(6t-1) = 96t^3 - 16t^2$

$$\int_C x^2 z\,ds = \int_0^1 (96t^3 - 16t^2)\sqrt{4^2+(-5)^2+6^2}\,dt$$

$$= \sqrt{77}\left(\tfrac{96}{4} - \tfrac{16}{3}\right) = \sqrt{77}\left(\tfrac{288-64}{12}\right) = \sqrt{77}\left(\tfrac{224}{12}\right) = \boxed{\tfrac{56\sqrt{77}}{3}}$$

§13.2#12 C parametrized by $r(t) = \langle t^2, t^3, t^2\rangle \quad 0 \le t \le 1$ thus $x = t^2$, $y = t^3$ and $z = t^2$ on C.

$$\int_C z\,dx + x\,dy + y\,dz \equiv \int_0^1 \left(t^2 \tfrac{dx}{dt}dt + t^2\tfrac{dy}{dt}dt + t^3\tfrac{dz}{dt}dt\right)$$

$$= \int_0^1 (2t^3 + 3t^4 + 2t^4)\,dt$$

$$= \frac{2}{4} + \frac{3}{5} + \frac{2}{5}$$

$$= \boxed{3/2}$$

§13.2 #14 $C = C_1 \cup C_2$ where C_1 goes from $(0,0,0)$ to $(1,2,-1)$ then C_2 goes from $(1,2,-1)$ to $(3,2,0)$.

$$C_1 : r_1(t) = (0,0,0) + t(1-0, 2-0, -1-0) = \langle t, 2t, -t \rangle, \quad 0 \le t \le 1$$

$$C_2 : r_2(\alpha) = (1,2,-1) + \alpha(3-1, 2-2, 0+1) = \langle 1+2\alpha, 2, \alpha-1 \rangle, \quad 0 \le \alpha \le 1$$

So we can break it into two parts,

$$\int_C x^2dx + y^2dy + z^2dz = \int_{C_1} x^2dx + y^2dy + z^2dz + \int_{C_2} x^2dx + y^2dy + z^2dz$$

On C_1 we have

$$\begin{array}{lll} x = t & dx/dt = 1 & x^2 = t^2 \\ y = 2t & dy/dt = 2 & y^2 = 4t^2 \\ z = -t & dz/dt = -1 & z^2 = t^2 \end{array}$$

Thus

$$\int_{C_1} x^2dx + y^2dy + z^2dz = \int_0^1 [t^2 + (4t^2)(2) + t^2(-1)]dt = \int_0^1 8t^2 = \boxed{\frac{8}{3}}.$$

On C_2 we have,

$$\begin{array}{lll} x = 1+2\alpha & dx/d\alpha = 2 & x^2 = 1 + 4\alpha + 4\alpha^2 \\ y = 2 & dy/d\alpha = 0 & y^2 = 4 \\ z = \alpha - 1 & dz/d\alpha = 1 & z^2 = \alpha^2 - 2\alpha + 1. \end{array}$$

Thus,

$$\begin{aligned} \int_{C_2} x^2dx + y^2dy + z^2dz &= \int_0^1 [(1+4\alpha+4\alpha^2)(2) + 4(0) + (\alpha^2 - 2\alpha + 1)]d\alpha \\ &= \int_0^1 (2 + 8\alpha + 8\alpha^2 + \alpha^2 - 2\alpha + 1)\, d\alpha \\ &= \int_0^1 (3 + 6\alpha + 9\alpha^2)\, d\alpha \\ &= 3 + \frac{6}{2} + \frac{9}{3} = 3 + 3 + 3 = \boxed{9}. \end{aligned}$$

Therefore, $\int_C x^2dx + y^2dy + z^2dz = \frac{8}{3} + 9 = \boxed{\frac{35}{3}}$.

<u>Remark</u>: You can use "t" for both C_1 and C_2. I introduce "α" to draw distinction between the cases.

$\boxed{\S 13.2 \# 18}$ Let $F = \langle yz, xz, xy \rangle$ and consider a curve C with parametrization $r(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 2$.

$$\int_C F \cdot dr \equiv \int_0^2 \left(F(r(t)) \cdot \frac{dr}{dt} \right) dt$$

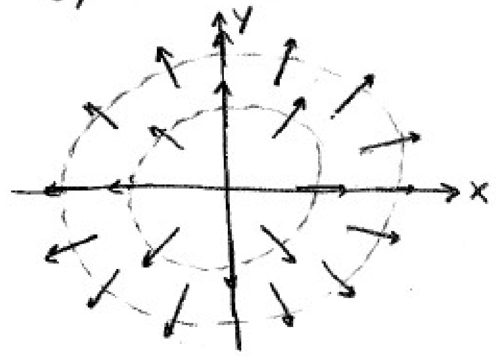
$$= \int_0^2 \langle t^5, t^4, t^3 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

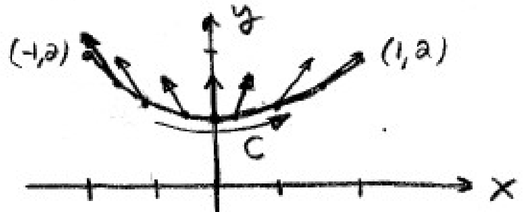
$$= \int_0^2 (t^5 + 2t^5 + 3t^5) dt$$

$$= \frac{6t^6}{6} \Big|_0^2 = 2^6 = \boxed{64}$$

$\boxed{\S 13.2 \# 22}$ Let $F = \langle x/r, y/r \rangle = \frac{1}{r} \langle x, y \rangle = \frac{1}{r} \vec{r} = \hat{r}$ where $r = \sqrt{x^2 + y^2}$. Analyze the line integral along C the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$.

Graphically, the vector field is purely radial,





looks like the tangents to the curve are equal and opposite comparing them to $\hat{r}$ on the left vs. right halves of C.

We may parametrize C by

$$r(t) = \langle t, 1 + t^2 \rangle, \; -1 \leq t \leq 1$$

Now calculate,

$$\int_C F \cdot dr \equiv \int_{-1}^1 F(t, 1+t^2) \cdot r'(t) \, dt$$

$$= \int_{-1}^1 \frac{1}{r} \langle t, 1+t^2 \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int_{-1}^1 \frac{1}{r} \left[t + (1+t^2) 2t \right] dt$$

$$= \int_{-1}^1 \underbrace{t \left(\frac{3+t^2}{\sqrt{t^2 + (1+t^2)^2}} \right)}_{\text{odd function!}} dt = 0.$$

§13.2#35 Find work done by $F = \langle y+z, x+z, x+y \rangle$ on a particle that moves from $(1,0,0)$ to $(3,4,2)$. Parametrize the path,

$C: r(t) = (1,0,0) + t(3-1, 4-0, 2-0) = \langle 2t+1, 4t, 2t \rangle \quad , 0 \le t \le 1.$

Calculate the work, by definition,

$$\int_C F \cdot dr = \int_0^1 \langle 6t, 4t+1, 6t+1 \rangle \cdot \langle 2, 4, 2 \rangle dt$$

$$= \int_0^1 (12t + 16t + 4 + 12t + 2) dt$$

$$= \int_0^1 (40t + 6) dt$$

$$= 20 + 6 = \boxed{26} \leftarrow \text{work done (ignoring units)}$$

<u>Remark</u>: notice $f = xy + xz + yz$ has $\nabla f = \langle y+z, x+z, x+y \rangle$ then $\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(3,4,2) - f(1,0,0) = 12 + 6 + 8 - 0 = \boxed{26}$.
It's easier to notice F is conservative and use the fundamental Th$^{\underline{m}}$ of line integrals from the next section.

§13.3#5 $F = \langle xe^y, ye^x \rangle \equiv \langle P, Q \rangle$ in the typical notation
notice $\partial_y P = xe^y$ yet $\partial_x Q = ye^x \quad \therefore \partial_y P \neq \partial_x Q \Rightarrow$ <u>F not conservative</u>
(Or we could equivalently argue $\nabla \times F \neq 0$ for $F = \langle xe^y, ye^x, 0 \rangle$
I mean this is afterall a special case of that more general argument)

§13.3#8 $F = \langle 1 + 2xy + \ln(x), x^2 \rangle$ note $\frac{\partial}{\partial y}(1 + 2xy + \ln(x)) = 2x = \frac{\partial}{\partial x}(x^2)$.
and for $x > 0$ we note F has continuous partial derivatives of comp. fncts.
Find then f such that $\nabla f = F = \langle \partial f/\partial x, \partial f/\partial y \rangle$

$$\frac{\partial f}{\partial y} = x^2 \Rightarrow f = \int x^2 \partial y = x^2 y + C_1(x)$$

(integral over y treats x as constant $\Rightarrow$ get a function of x as the "constant" for the $\int$)

$$\frac{\partial f}{\partial x} = 1 + 2xy + \ln(x) = \frac{\partial}{\partial x}(x^2 y + C_1) = 2xy + \frac{dC_1}{dx}$$

$$\therefore \frac{dC_1}{dx} = 1 + \ln(x) \quad \text{note} \quad \int \underbrace{\ln(x)}_{u} \underbrace{dx}_{dv} = \underbrace{x\ln(x) - \int x \frac{dx}{x}}_{\text{I.B.P.}} = x\ln(x) - x + C.$$

$$\Rightarrow C_1 = \int (1 + \ln(x)) dx = x + x\ln(x) - x + C_2 \quad \therefore \boxed{f = x^2 y + x\ln(x) + C_2}$$

§13.3#12 $F = \langle y, x+2y \rangle$ we can deduce by inspection that $f = xy + y^2$ has $\nabla f = \langle y, x+2y \rangle$. Let C be the semi-circle starting at $(0,1)$ going to $(2,1)$

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(2,1) - f(0,1) = (2+1) - (1) = \boxed{2}.$$

§13.3#16 Let C go from $(0,1,-1)$ to $(1,2,1)$. Find $\int_C F \cdot dr$ for $F = \langle 2xz + y^2, 2xy, x^2 + 3z^2 \rangle$. Find the potential fnct. φ with $\nabla\varphi = F$,

$$\frac{\partial \varphi}{\partial x} = 2xz + y^2 \Rightarrow \varphi = x^2 z + xy^2 + C_1(y,z)$$

$$\frac{\partial \varphi}{\partial y} = 2xy = 2xy + \frac{\partial C_1}{\partial y} \Rightarrow \frac{\partial C_1}{\partial y} = 0 \quad \text{thus} \quad C_1 = C_1(z)$$

(**** allow me this abuse of notation **** See below.)

$$\frac{\partial \varphi}{\partial z} = x^2 + 3z^2 = \frac{\partial}{\partial z}\left(x^2 z + xy^2 + C_1\right) = x^2 + \frac{\partial C_1}{\partial z} = x^2 + \frac{dC_1}{dz}.$$

$$\Rightarrow \frac{dC_1}{dz} = 3z^2 \therefore C_1 = z^3 \Rightarrow \underline{\varphi = x^2 z + xy^2 + z^3}$$

Now use FTC for line integrals,

$$\int_C F \cdot dr = \int_C (\nabla \varphi) \cdot dr = \varphi(1,2,1) - \varphi(0,1,-1) = (1+4+1) - (-1)^3 = 7$$

§13.3#22 Let $F = \langle e^{-y}, -xe^{-y} \rangle$ find work done by F on an object that moves from $(0,1)$ to $(2,0)$. Its not hard to see that $\varphi = xe^{-y}$ has $\nabla \varphi = F$. Thus

$$W = \int_C F \cdot dr = \int (\nabla \varphi) \cdot dr = \varphi(2,0) - \varphi(0,1) = 2 - 0 = \boxed{2}$$

Where C is a path from $(0,1)$ to $(2,0)$, it matters not which path in particular, because F is conservative the work done depends only on the total displacement.

Remark: I said that $C_1 = C_1(z)$ is an abuse of notation because strictly speaking C_1 is a function whereas $C_1(z)$ is the value of the function at z. I do think $C_1 = C_1(z)$ or $f = f(x)$ or $g = g(x,y)$ is a useful shorthand for saying C_1 is a fnct. of z or f a fnct of x or g a fnct. of x & y.