SOLVING SYSTEMS OF LINEAR EQUATIONS

Our ultimate goal is to solve systems of ODEs, but first we should find how to solve the corresponding algebra problem. To do this well, we will introduce the notion of matrices to make our work more understandable.

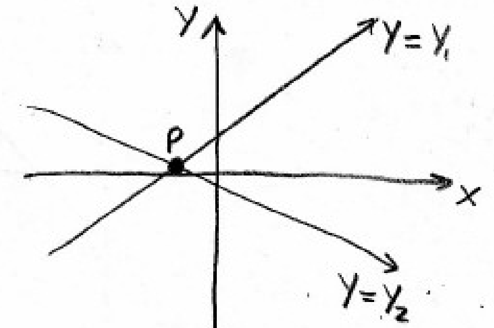
2 EQUATIONS AND 2 UNKNOWNS

$ax + by = f$	: Eq^{n}(1)
$cx + dy = g$	: Eq^{n}(2)

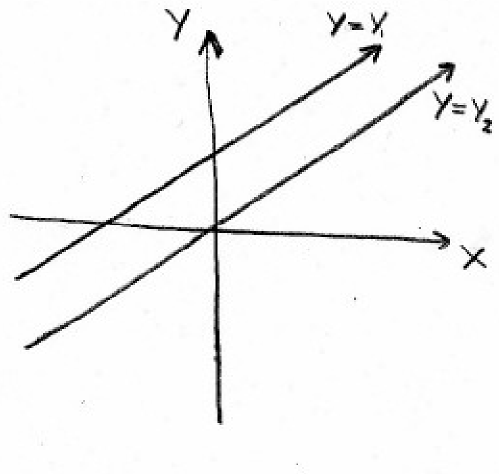
We are thinking of a, b, c, d, f, g as being given but arbitrary real numbers. What are the possibilities?

$$Y = f/b - (a/b)x \qquad (b \neq 0)$$
$$Y = g/d - (c/d)x \qquad (d \neq 0)$$

We can draw the possibilities in graphical form



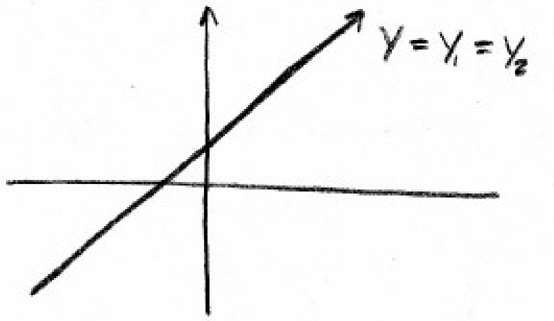
- lines with differing slopes intersect somewhere, the intersection is where <u>both</u> eq^{n}'s are true. Thus the intersection point P is the sol^{n} to the system of eq^{n}'s.



- lines with the same slope but different Y-intercepts never intersect. Thus no intersection, thus the corresponding system of eq^{n}'s has no sol^{n}'s.

2 EQUATIONS AND 2 UNKNOWNS CONTINUED

One more possibility exists



- the two lines are in fact the same line. In this case each point on the line is an intersection point, thus there are ∞ly many sol^n.

To summarize, given a system of 2 eq^n's with 2 unknowns (X & Y) there are 3 distinct possibilities,

1.) unique sol^n.
2.) no sol^n.
3.) ∞ly many sol^n.

I'll now illustrate three methods to solve systems of eq^n's:

$\boxed{E1}$

$$\begin{aligned} X + Y &= 3 \\ +\ (X - Y &= -1) \\ \hline 2X &= 2 \end{aligned}$$

$$2X = 2 \Rightarrow X = 1 \quad \& \quad Y = 3 - X = 3 - 1 = 2 = Y$$

We are in case 1.) and we solved it by adding eq^n's to eliminate the variable Y. This general strategy is called <u>Gauss - Jordan elimination</u>. (See text for a more precise statement of the algorithm.)

$\boxed{E2}$

$$\begin{aligned} X + Y &= 3 \\ X - Y &= -1 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Observe $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \dfrac{1}{-1-1}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$ is the inverse matrix

$$\underbrace{\cancel{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}}_{I}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 - 1/2 \\ 3/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Thus reading off components gives $\boxed{\begin{aligned} X &= 1 \\ Y &= 2 \end{aligned}}$

$\boxed{E3}$ $\begin{array}{l} x+y=3 \\ x-y=-1 \end{array} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \leftrightarrow A\vec{x} = \vec{b}$

$$x = \frac{\begin{vmatrix} 3 & 1 \\ -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-3+1}{-1-1} = \frac{-2}{-2} = \boxed{1 = x}$$

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-1-3}{-1-1} = \frac{-4}{-2} = \boxed{2 = y}$$

This is known as Kramer's Rule. The $|\ |$'s denote the determinant of the matrix.

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We have solved a particular example 3 unique ways. Let me summarize,

1.) $\boxed{E1}$ Gauss Jordan Elimination
2.) $\boxed{E2}$ $Ax = b \Rightarrow x = A^{-1}b$; using inverse matrix to solve system (I don't know a name for it ¨)
3.) $\boxed{E3}$ Kramer's Rule

In 2.) and 3.) we encountered the determinant of the coefficient matrix ; $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$. Notice if this were zero then neither of the last two methods could be employed. On the other hand Gauss Jordan Elimination would still work. We use all three methods in this course, but Gauss Jordan must be used when the determinant is zero. Gauss Jordon Elimination is where we add/substract and multiply by $\neq 0$ # the eq$^{\underline{o}}$'s in a system towards the goal of uncoupling the eq$^{\underline{o}}$'s. This is also known as elimination by substition and the calculator will accomplish it using rref or ref on a TI-85, 86, 89, 92, ..