Laplace Transforms

In our treatment of constant coefficient differential eq^{n}s we discovered that we could translate the problem of calculus to a corresponding problem of algebra. Laplace transforms do something similar, however Laplace transforms allow us to solve a wider class of problems. In particular the Laplace transform will allow us an elegant sol^n to problems that have discontinuous forcing functions ($g(x)$ is the "forcing function"). In short, Laplace transforms provide a powerful method to solve a wide class of ODE's which appear in common applications (especially electrical engineering where t = time & s = frequency)

Def^n/ Let $f(t)$ be a function with $\text{dom}(f) = [0, \infty)$. The Laplace transform of f is the function F defined by

$$F(s) \equiv \int_0^\infty e^{-st} f(t)\,dt \equiv \mathcal{L}\{f\}(s)$$

The $\text{dom}(F)$ is chosen to be all $s \in \mathbb{R}$ for which the integral exists.

Remark: You should recall that $\int_0^\infty g(t)\,dt \equiv \lim_{N\to\infty} \int_0^N g(t)\,dt$. In this course we will usually write the limit explicitly,

(explicit) $$\int_0^\infty e^{-x}\,dx = \lim_{N\to\infty} \int_0^N e^{-x}\,dx = \lim_{N\to\infty} \left(-e^{-N} + 1\right) = 1$$

(implicit) $$\int_0^\infty e^{-x}\,dx = e^{-x}\Big|_0^\infty = -e^{-\infty} + 1 = 0 + 1 = 1$$

if I ask you to be explicit then follow the direction, you will likely see the implicit version in applied courses. The implicit version is usually ok, but when something subtle arises it will confuse or disguise this issue. For example $\infty e^{-\infty} = ?$.

Th^m(1) The Laplace transform is a linear operator

$$\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$$
$$\mathcal{L}\{cf\} = c\,\mathcal{L}\{f\}$$

Proof: follows imediately from linearity of integral.

Standard Examples of $\mathcal{L}$

E1 Calculate $\mathcal{L}\{f\}$ for constant function $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$
$$= \int_0^\infty e^{-st}\,dt$$
$$= \frac{-1}{s} e^{-st}\Big|_0^\infty$$
$$= \frac{-1}{s}\left(\cancel{e^{-s\infty}}^{\,0} - 1\right) \qquad \text{provided } s > 0.$$
$$= \frac{1}{s} \quad \text{for } s > 0. \qquad \therefore \boxed{\mathcal{L}\{1\}(s) = \frac{1}{s}}$$

E2 Find $\mathcal{L}\{e^{at}\}$. Let $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{-st} e^{at}\,dt$$
$$= \int_0^\infty e^{-(s-a)t}\,dt$$
$$= \frac{-1}{s-a} e^{-(s-a)t}\Big|_0^\infty$$
$$= \frac{-1}{s-a}\left(e^{-(s-a)\infty} - 1\right)$$
$$= \frac{1}{s-a} \quad \text{for } s > a \qquad \therefore \boxed{\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}}$$

E3 for $b \neq 0$ calculate $\mathcal{L}\{\sin bt\}$. Let $f(t) = \sin bt$,

$$F(s) = \int_0^\infty e^{-st} \sin(bt)\,dt$$
$$= \int_0^\infty e^{-st} \frac{1}{2i}\left(e^{ibt} - e^{-ibt}\right) dt$$
$$= \int_0^\infty \frac{1}{2i}\left(e^{(ib-s)t} - e^{(-ib-s)t}\right) dt$$
$$= \frac{1}{2i}\left[\frac{1}{ib-s} e^{(ib-s)t} - \frac{1}{(-ib-s)} e^{(-ib-s)t}\right|_0^\infty$$
$$= \frac{1}{2i}\left[\frac{-s-ib}{s^2+b^2} e^{(ib-s)t} - \frac{-s+ib}{s^2+b^2} e^{(-ib-s)t}\right|_0^\infty$$
$$= \frac{1}{s^2+b^2}\left\{-se^{-st} \underbrace{\frac{1}{2i}\left(e^{ibt} - e^{-ibt}\right)}_{\sin(bt)} - be^{-st} \underbrace{\frac{1}{2}\left(e^{ibt} + e^{-ibt}\right)}_{\cos(bt)}\right\}\Big|_0^\infty$$

Remark: I'm calculating this integral in a slightly unconventional manner, perhaps some of you will find it useful.

E3 Continued,

$$F(s) = \lim_{N\to\infty} \left\{ \frac{-1}{s^2+b^2} e^{-st}\left[s\sin(bt) + b\cos(bt)\right] \Big|_0^N \right.$$

$$= \lim_{N\to\infty} \left\{ \frac{-1}{s^2+b^2} \underbrace{e^{-sN}}_{\text{goes to zero}} \underbrace{\left[s\sin(bN) + b\cos(bN)\right]}_{\text{bounded as } N\to\infty} + \frac{b}{s^2+b^2} \right\} = \frac{b}{s^2+b^2}$$

- Could prove the first term $\to 0$ using $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$ plus the squeeze Th^{m}. Anyway we found

$$\boxed{\mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2+b^2}}$$

E4 $f(t) = \begin{cases} 0 & 0 \leq t < a \\ 1 & a < t \end{cases}$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\,dt$$

$$= \frac{-1}{s}\left(e^{-s\infty} - e^{-sa}\right)$$

$$= e^{-sa}\,\frac{1}{s}$$

This function $f(t) = H(t-a)$ is an example of a Heaviside or unit-step function. Notice it is discontinous at $t = a$.

Def^{n} of piecewise continuous

<u>Remark</u>: You may recall that <u>any</u> continous function is integrable. In fact, it is possible to integrate any <u>function with finitely many jump-discontinuities</u>. You just break up the integral into pieces, the value of the function at the discontinuities is irrelevant.

Def^{n}/ jump-discontinuity for a function f is some point where the left & right limits of f are finite yet do not agree.

When Does Laplace Transform Work?

We need piecewise continuity at a minimum, in addition we need $f(t)$ to not grow to fast or else the integral in the Laplace transform will diverge.

Def^n/ A function $f(t)$ is said to be of exponential order α if $\exists$ positive constants $T, M > 0$ such that
$$|f(t)| \leq Me^{\alpha t} \quad \text{for all } t \geq T$$

This criteria allows us to state when the Laplace Transform of $f(t)$ exists (that is when the $\int_0^\infty$ converges)

Th^m (2) If $f(t)$ is piecewise continous on $[0,\infty)$ and of exponential order α the $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$

Proof:
$$\begin{aligned}
\int_0^\infty e^{-st} f(t)\,dt &\leq \int_0^\infty e^{-st}|f(t)|\,dt \\
&\leq \underbrace{\int_0^T e^{-st}|f(t)|\,dt} + \int_T^\infty e^{-st} Me^{\alpha t}\,dt \\
&\leq C_1 + Me^{-(s-\alpha)t}\Big|_T^\infty \\
&\leq C_1 - \frac{M}{s-\alpha}\left(e^{-(s-\alpha)\infty} - e^{-(s-\alpha)T}\right)
\end{aligned}$$
$e^{-(s-\alpha)\infty} \to 0$ for $s > \alpha$

Thus $\mathcal{L}\{f\}(s) < \infty$ when $s > \alpha$.
(See text for a better proof on pg. 358)

Examples of Laplace Transformable functions

- e^{at} has exponential order a.
- $\sin t$ has $|\sin t| \leq 1 = e^{0\cdot t} \Rightarrow \sin t$ of exp. order zero.
- $\cos bt$ is of exponential order zero.
- $e^{at}\sin t$ is of exp. order a.
- t^n has $|t^n| < e^t$ for $t > 1 \Rightarrow t^n$ is of exp. order one.

So all the functions that appear as fundamental sol$^=$ of const. coeff. O.D.E.'s can be Laplace transformed. This is good as it is necessary if $\mathcal{L}$ is to do common examples.

Known Laplace Transforms

Table 7.1 pg. 358

$f(t)$	$\mathcal{L}\{f\}(s) \equiv F(s)$	dom(F)
1	$\frac{1}{s}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$t^n,\ n=1,2,\dots$	$\frac{n!}{s^{n+1}}$	$s>0$
$\sin bt$	$\frac{b}{s^2+b^2}$	$s>0$
$\cos bt$	$\frac{s}{s^2+b^2}$	$s>0$
$e^{at}t^n,\ n=1,2,\dots$	$\frac{n!}{(s-a)^{n+1}}$	$s>a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$

These can be calculated directly from the definition

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

I'll delay the proof of several of these till we know more.

these are redundant in view of a later Th^{m}, but we'll take them as knowns for convenience now.

E5 $f(t) = t^2 - 3t - 2e^{-t}\sin 3t$

$$F(s) = \mathcal{L}\{t^2\}(s) - 3\mathcal{L}\{t\}(s) - 2\mathcal{L}\{e^{-t}\sin 3t\}(s)$$
$$= \boxed{\frac{2}{s^3} - \frac{3}{s^2} - 2\left(\frac{3}{(s+1)^2+9}\right)}$$

E6 $f(t) = e^{-2t}\cos\sqrt{3}t - t^2 e^{-2t}$

$$F(s) = \mathcal{L}\{e^{-2t}\cos(\sqrt{3}t)\}(s) - \mathcal{L}\{t^2 e^{-2t}\}(s)$$
$$= \boxed{\frac{s+2}{(s+2)^2+3} - \frac{2}{(s+2)^3}}$$

Remark: taking the Laplace transform with the help of the table is not bad. The trouble comes later when we try to go backwards.

$\boxed{\text{Th}^{m}(3)}$ (Shift Theorem) If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$ then for $s > \alpha + a$

$$\mathcal{L}\{e^{at} f(t)\}(s) = F(s-a)$$

Proof:

$$\mathcal{L}\{e^{at} f(t)\}(s) = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$
$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$
$$= F(s-a).$$

E7 $\mathcal{L}\{e^{at}\sin bt\}(s) = F(s-a) = \boxed{\dfrac{b}{(s-a)^2 + b^2}}$ for $f(t) = \sin bt$, $F(s) = \dfrac{b}{s^2+b^2}$

E8 $\mathcal{L}\{e^{at}\}(s) = F(s-a) = \boxed{\dfrac{1}{s-a}}$ for $f(t) = 1$, $F(s) = 1/s$

Th^{m} (4) (Laplace Transform of Derivative). Let f and f' be piecewise continuous with exponential order α then for $s > \alpha$,

$$\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0)$$

Proof:

$$\mathcal{L}\{f'\}(s) = \lim_{N\to\infty} \int_0^N e^{-st} \frac{d}{dt}(f(t))\,dt$$
$$= \lim_{N\to\infty} \int_0^N \left[\frac{d}{dt}\left(e^{-st} f(t)\right) - \frac{d}{dt}\left(e^{-st}\right) f(t)\right] dt$$
$$= \lim_{N\to\infty} \left(e^{-sN} f(N) - f(0) + \int_0^N s\, e^{-st} f(t)\,dt\right)$$
$$= \lim_{N\to\infty} \left(e^{-sN} f(N)\right) - f(0) + s \lim_{N\to\infty} \int_0^N e^{-st} f(t)\,dt$$
$$= -f(0) + s\,\mathcal{L}\{f\}(s).$$

Where we noted since for $t > \alpha$ we have $|f(t)| \leq Me^{\alpha t}$ thus $|e^{-sN} f(N)| \leq e^{-sN} M e^{\alpha N} = M e^{N(\alpha - s)} \longrightarrow 0$ as $N \to \infty$ provided $s > \alpha$.

We can derive similar formulas for higher derivatives,

$$\begin{aligned}\mathcal{L}\{f''\}(s) &= s\,\mathcal{L}\{f'\}(s) - f'(0) \qquad \text{using Th}^{m}(4).\\ &= s\,(s\,\mathcal{L}\{f\}(s) - f(0)) - f'(0)\\ &= s^2\,\mathcal{L}\{f\} - sf(0) - f'(0).\end{aligned}$$

Th^{m}(5) Let $f, f', \dots, f^{(n-1)}$ be continous and $f^{(n)}$ piecewise continuous all of them of exponential order α. Then for $s > \alpha$,

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\,\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Proof: follows from calculation similar to the one above Th^{m}(5).

E9 Lets use Th^{m}(4) and Th^{m}(5) to transform the constant coefficient DEq^{n} $aY'' + bY' + cY = 0$ where $Y' = \frac{dY}{dt}$

$$\begin{aligned}\mathcal{L}\{Y''\}(s) &= s^2\,\mathcal{L}\{Y\}(s) - sY(0) - Y'(0) = s^2\,\mathrm{Y}(s) - sY(0) - Y'(0)\\ \mathcal{L}\{Y'\}(s) &= s\,\mathcal{L}\{Y\}(s) - Y(0) = s\,\mathrm{Y}(s) - Y(0)\\ \mathcal{L}\{Y\}(s) &= \mathrm{Y}(s)\end{aligned}$$

It is customary to use lower case Y for $Y(t)$ then uppercase Y to denote the Laplace transform $\mathcal{L}\{Y\}(s) \equiv \mathrm{Y}(s)$. Taking the Laplace transform of $aY'' + bY' + cY = 0$ yields

$$a\left(s^2\mathrm{Y} - sY(0) - Y'(0)\right) + b\left(s\mathrm{Y} - Y(0)\right) + c\,\mathrm{Y} = 0$$

$$(as^2 + bs + c)\,\mathrm{Y} = asY(0) + aY'(0) + bY(0)$$

Notice we have change a differential eq^{n} in t to an algebraic eq^{n} in s. We'll come back to this example later.

E10 Let $g(t) = \int_0^t f(u)\,du$ then $g'(t) = f(t)$ by FTC. Thus

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \mathcal{L}\{g'\}(s)\\ &= s\,\mathcal{L}\{g\}(s) - g(0)\\ &= s\,G(s) - \cancel{\int_0^0 f(u)\,du} \to 0\end{aligned}$$

$$\therefore \boxed{\frac{1}{s}\mathcal{L}\{f\}(s) = \mathcal{L}\left\{\int_0^t f(u)\,du\right\}(s)}$$

Remark: In E9 and E10 we observe that the Laplace transform changes differentiation w.r.t. t into multiplication by s, and some extra stuff that relates to the initial conditions.

Th^m(6) Let $F(s) = \mathcal{L}\{f\}(s)$ and assume $f(t)$ is piecewise continous on $[0,\infty)$ and of exponential order α. Then for $s>\alpha$

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)$$

"Proof": We'll see why $n=1$ works,

$$\begin{aligned}\frac{dF}{ds} &= \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right) \\ &= \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right) dt \\ &= \int_0^\infty -t e^{-st} f(t)\,dt \\ &= (-1)^1 \int_0^\infty e^{-st}\, t f(t)\,dt \\ &= (-1)^1 \mathcal{L}\{t f(t)\}(s)\end{aligned}$$

nontrivial step, serious proof that this is allowed can take a page of careful reasoning. However, intuitively it's reasonable.

Then next try $n=2$,

$$\begin{aligned}\frac{d^2F}{ds^2} = \frac{d}{ds}\left(\frac{dF}{ds}\right) &= (-1)^1 \frac{d}{ds}\int_0^\infty e^{-st}\, t f(t)\,dt \\ &= (-1)^1 \int_0^\infty \frac{d}{ds}\left(e^{-st}\, t f(t)\right) dt \\ &= (-1)^2 \int_0^\infty e^{-st}\, t^2 f(t)\,dt \\ &= (-1)^2 \mathcal{L}\{t^2 f(t)\}(s).\end{aligned}$$

It's clear we'll find that $\frac{d^nF}{ds^n} = (-1)^n \mathcal{L}\{t^n f(t)\}(s)$ which upon multiplication by $(-1)^n$ is exactly the Th^m.

Properties of Laplace Transforms Summary (table 7.2)

$$\boxed{\begin{array}{l} \mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\} \\ \mathcal{L}\{cf\} = c\,\mathcal{L}\{f\} \\ \mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a) \\ \mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0) \\ \mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0) \\ \mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \\ \mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s) \quad \text{where } \mathcal{L}\{f\} = F. \end{array}}$$

E11 Let $f(t) = (1+e^{-t})^2 = 1 + 2e^{-t} + e^{-2t}$ then

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{1\}(s) + 2\mathcal{L}\{e^{-t}\}(s) + \mathcal{L}\{e^{-2t}\}(s)$$
$$= \boxed{\frac{1}{s} + \frac{2}{s+1} + \frac{1}{s+2}}$$

E12 Let $f(t) = te^{2t}\cos 5t$

$$\mathcal{L}\{f\}(s) = -\frac{d}{ds}\left(\mathcal{L}\{e^{2t}\cos 5t\}(s)\right) \text{, using Th}^{m}(6)\ n=1$$
$$= -\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+25}\right) \text{, table 7.1.}$$
$$= -\frac{(s-2)^2+25-(s-2)[2(s-2)]}{[(s-2)^2+25]^2} \text{, quotient rule.}$$
$$= \boxed{\frac{(s-2)^2-25}{[(s-2)^2+25]^2} = F(s)}$$

E13 Let $f(t) = t\sin^2 t$

$$\mathcal{L}\{f\}(s) = -\frac{d}{ds}\left(\mathcal{L}\{\sin^2 t\}(s)\right)$$
$$= -\frac{d}{ds}\left(\mathcal{L}\{\tfrac{1}{2}(1-\cos 2t)\}(s)\right) \quad \text{trig. identity you should know.}$$
$$= -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$
$$= -\frac{1}{2}\left(\frac{-1}{s^2} - \frac{s^2+4-s(2s)}{(s^2+4)^2}\right)$$
$$= \boxed{\frac{1}{2}\left(\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right) = F(s) = \mathcal{L}\{f\}(s)}$$